A counterexample to a conjecture of Grünbaum on piercing convex sets in the plane

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July 22, 2013

Abstract

A collection of sets $\mathcal{F}$ has the (p, q)-property if out of every p elements of $\mathcal{F}$ there are q that have a point in common. A transversal of a collection of sets $\mathcal{F}$ is a set A that intersects every member of $\mathcal{F}$. Grünbaum conjectured that every family $\mathcal{F}$ of closed, convex sets in the plane with the $(4, 3)$-property and at least two elements that are compact has a transversal of bounded cardinality. Here we construct a counterexample to his conjecture. On the positive side, we also show that if such a collection $\mathcal{F}$ contains two *disjoint* compacta then there is a transveral of cardinality at most 13.

1 Introduction and statement of results

Let $\mathcal{F}$ be a collection of sets. A *transversal* of $\mathcal{F}$ is a set A that intersects every member of $\mathcal{F}$ (that is, $A \cap F \neq \emptyset$ for all $F \in \mathcal{F}$). The *transversal number* of *piercing number* $\tau(\mathcal{F})$ of $\mathcal{F}$ is the smallest size of a transversal, i.e.

$$\tau(\mathcal{F}) := \min_{A \text{ transversal of } \mathcal{F}} |A|.$$

(Note that $\tau(\mathcal{F}) = \infty$ if no finite transversal exists.)

A collection of sets $\mathcal{F}$ has the (p, q)-property if out of every p sets of $\mathcal{F}$ there are q that have a point in common. In 1957, Hadwiger and Debrunner [2] conjectured that for every d and every $p \geq q \geq d + 1$ there is a universal constant $c = c(d; p, q)$ such that every finite collection $\mathcal{F}$ of convex sets in $\mathbb{R}^d$ with the (p, q)-property satisfies $\tau(\mathcal{F}) \leq c$. (By considering hyperplanes in general position it is easily seen that for $q \leq d$ no such universal constant c can exist.) Many years later, in 1992, Alon and Kleitman [1] finally proved the conjecture of Hadwiger and Debrunner by cleverly combining various pre-existing tools from the literature.

In the special case when $p = q = d + 1$ the Hadwiger-Debrunner conjecture reduces to the classical theorem of Helly [3] which states that if $\mathcal{F}$ is a finite family of convex sets in $\mathbb{R}^d$ such that every $d + 1$ members of $\mathcal{F}$ have a point in common then $\tau(\mathcal{F}) = 1$. A variant of Helly's theorem states that if $\mathcal{F}$ is an infinite collection of closed, convex sets in $\mathbb{R}^d$ and at least one member of $\mathcal{F}$ is compact then $\tau(\mathcal{F}) = 1$.

Erdős conjectured that in the first nontrivial case of the Hadwiger-Debrunner problem, a similar variant would be true. That is, he conjectured that if $\mathcal{F}$ is a collection of closed, convex sets in the plane with the $(4, 3)$-property and one of the members of $\mathcal{F}$ is compact, then $\tau(\mathcal{F}) \leq c$ for some universal constant c. Boltyanski and Soifer included this conjecture in the first edition of their book "Geometric Etudes in Combinatorial Mathematics" and they offered a prize of $25 for its solution. Eighteen years later, Grünbaum found a counterexample while proofreading the second edition, earning the reward. Grünbaum also made a conjecture of his own, stating that if $\mathcal{F}$ is a collection of closed, convex sets in the plane, and *two* members of $\mathcal{F}$ are compact then $\tau(\mathcal{F})$ is finite. (See [5], pages 198-199.) Here we show that Grünbaum's conjecture fails as well:

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Theorem 1 *There exists a collection $\mathcal{F}$ of closed, convex subsets of the plane such that*

(i) *$\mathcal{F}$ has the $(4,3)$-property, and;*

(ii) *Two of the elements of $\mathcal{F}$ are compact, and;*

(iii) *$\tau(\mathcal{F}) = \infty$.*

On the positive side, we show that any collection $\mathcal{F}$ of closed, convex sets in the plane that contains two disjoint compacta and satifies the $(4,3)$-property does have universally bounded transversal number:

Theorem 2 *If $\mathcal{F}$ is a collection of closed, convex sets in the plane such that*

(i) *$\mathcal{F}$ has the $(4,3)$-property, and;*

(ii) *$\mathcal{F}$ contains two disjoint compacta,*

then $\tau(\mathcal{F}) \leq 13$.

2 The counterexample

Let us set

$$F_1 := [-1,1] \times \{0\}, \quad F_2 := [0,2] \times \{0\}.$$

Let $t_1 < t_2 < t_3 < \dots$ be a strictly increasing sequence of numbers between 0 and 1, and let $s_1 > s_2 > \dots$ be a strictly decreasing sequence of negative numbers that tends to $-\infty$. (For instance $t_n := 1 - \frac{1}{n}, s_n := -n$ would be a valid choice.) Set $p_n := (t_n, 0)$; let ℓ_n denote the vertical line through p_n; and let ℓ'_n denote the line through p_n of slope s_n.

For $n \geq 3$ we now let F_n be the set of all points either on or to the left of ℓ_n and either on or above ℓ'_n. See figure 1.

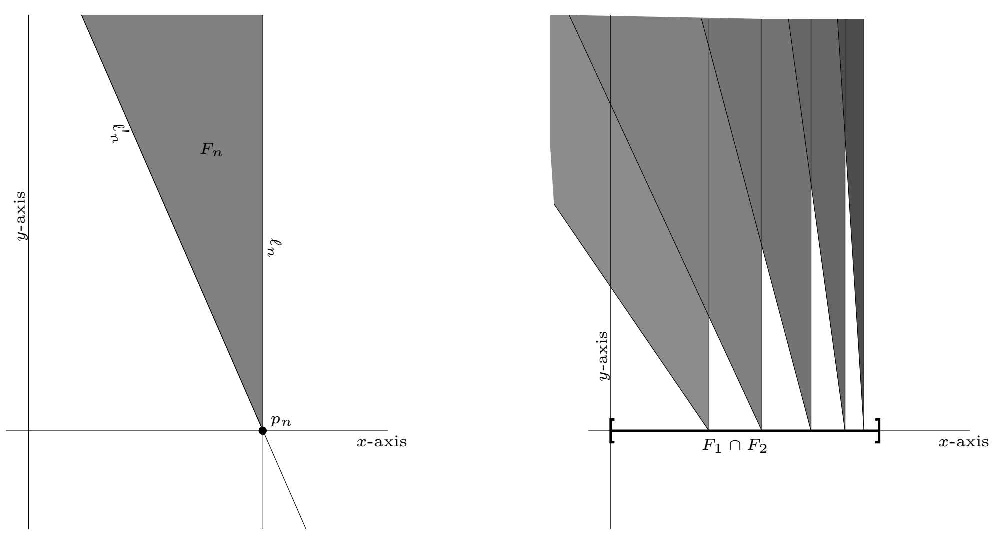

Figure 1: The construction of F_n for $n \geq 3$ (left) and part of the collection $\mathcal{F}$ (right).

Observe that by construction F_n contains all sufficiently high points on the y-axis for all $n \geq 3$:

$$\text{For each } n \geq 3 \text{ there exists a } y_n > 0 \text{ such that } \{(0,y) : y \geq y_n\} \subseteq F_n. \tag{1}$$

Let $\mathcal{F} := \{F_1, F_2, \dots\}$ be the resulting infinite collection of closed convex sets. We first establish that $\mathcal{F}$ has the $(4,3)$-property.

Lemma 3 *$\mathcal{F}$ has the* $(4,3)$*-property.*

Proof: Let us pick four abitrary distinct indices $i_1 < i_2 < i_3 < i_4$ and consider the quadruple $F_{i_1}, F_{i_2}, F_{i_3}, F_{i_4} \in \mathcal{F}$.

If $i_1 = 1$ and $i_2 = 2$ then clearly $F_{i_1} \cap F_{i_2} \cap F_{i_3} = \{p_{i_3}\}$, so that $F_{i_1}, F_{i_2}, F_{i_3}$ is an intersecting triple. We can thus assume that $i_2, i_3, i_4 > 2$. In this case $F_{i_2} \cap F_{i_3} \cap F_{i_4} \neq \emptyset$ by the observation (1). ∎

It remains to show that $\mathcal{F}$ does not have a finite transversal.

Lemma 4 $\tau(\mathcal{F}) = \infty$.

Proof: It suffices to show that every point of the plane is in finitely many elements of $\mathcal{F}$. Let $a = (a_x, a_y) \in \mathbb{R}^2$ be arbitrary. If $a_y \leq 0$ then a is in at most three elements of $\mathcal{F}$. Let us therefore assume $a_y > 0$. In this case, if $a_x \geq t_n$ for all $n \in \mathbb{N}$ then a is in no element of $\mathcal{F}$. Let us therefore assume that there is at least one $n \in \mathbb{N}$ such that $a_x < t_n$. Let us fix an n_0 such that $a_x < t_{n_0}$, and set

$$s := -\frac{a_y}{t_{n_0} - a_x}.$$

(Note that s is exactly the slope of the line through a and p_{n_0}.)
Since $s_n \to -\infty$, there is an m_0 such that $s_n < s$ for all $n \geq m_0$.

Observe that for all $n \geq \max(n_0, m_0)$ the point a is below the line ℓ'_n (as the point p_n is to the right of p_{n_0} and ℓ'_n has a steeper slope than s). This shows that $a \notin F_n$ for all $n \geq \max(n_0, m_0)$. Hence a is in finitely many elements of $\mathcal{F}$ as required. ∎

Remark: By adding additional compact sets to $\mathcal{F}$ that each contain $[0,1] \times \{0\}$ we can obtain a collection $\mathcal{F}'$ that contains an arbitrary number of compacta, and still has the $(4,3)$-property and $\tau(\mathcal{F}') = \infty$.

3 The proof of Theorem 2

The proof of the Hadwiger-Debrunner conjecture by Alon and Kleitman [1] does not give a good bound on the universal constant c. A better bound on this constant for the special case when $p = 4, q = 3$ was later given by Kleitman, Gyarfas and Tóth [4].

Theorem 5 (Kleitman et al. [4]) *If $\mathcal{F}$ is a finite collection of convex sets in the plane with the* $(4,3)$*-property then* $\tau(\mathcal{F}) \leq 13$.

A standard compactness argument (which we do not repeat here) shows that the same also holds if $\mathcal{F}$ is an infinite collection of convex compacta with the $(4,3)$-property.

Corollary 6 *If $\mathcal{F}$ is an infinite collection of convex, compact sets in the plane and $\mathcal{F}$ has the* $(4,3)$*-property then* $\tau(\mathcal{F}) \leq 13$.

Proof of Theorem 2: Let $\mathcal{F}$ be an arbitrary infinite collection of closed, convex sets with the $(4,3)$-property with two sets $A, B \in \mathcal{F}$ that are disjoint and compact. Let us set

$$F_0 := \text{conv}(A \cup B).$$

Let us first observe that

$$F \cap F_0 \neq \emptyset \text{ for all } F \in \mathcal{F}. \tag{2}$$

To see this, suppose that some $F \in \mathcal{F}$ is disjoint from F_0, and let $F' \in \mathcal{F}$ be an arbitrary element distinct from F, F_1, F_2 and F_0. Then the quadruple F_1, F_2, F, F' does not have an intersecting triple as every triple contains a pair of disjoint sets. But this contradicts the $(4,3)$-property! Hence (2) holds as claimed.

Next, we claim that

$$\text{If } F_1, F_2, F_3 \in \mathcal{F} \text{ are such that } F_1 \cap F_2 \cap F_3 \neq \emptyset \text{ then also } F_0 \cap F_1 \cap F_2 \cap F_3 \neq \emptyset. \qquad (3)$$

To see that the claim (3) holds, consider an arbitrary triple $F_1, F_2, F_3 \in \mathcal{F}$ such that $F_1 \cap F_2 \cap F_3 \neq \emptyset$. Let us assume $F_1 \cap F_2 \cap F_3 \not\subseteq F_0$ (otherwise we are done), and fix a $q \in (F_1 \cap F_2 \cap F_3) \setminus F_0$. By considering the quadruple A, B, F_1, F_2 we see that we either have $A \cap F_1 \cap F_2 \neq \emptyset$ or $B \cap F_1 \cap F_2 \neq \emptyset$. In either case, there is a point $p_{12} \in F_0 \cap F_1 \cap F_2$. Similarly there are points $p_{13} \in F_0 \cap F_1 \cap F_2, p_{23} \in F_0 \cap F_2 \cap F_3$.

By Radon's lemma the set $\{q, p_{12}, p_{13}, p_{23}\}$ can be partitioned into two sets whose convex hulls intersect. Note that we cannot have that $q \in \text{conv}(\{p_{12}, p_{13}, p_{23}\})$ since $q \notin F_0$ and $p_{12}, p_{13}, p_{23} \in F_0$ and F_0 is convex. Hence, up to relabelling of the indices we have either $p_{23} \in \text{conv}(\{q, p_{12}, p_{13}\})$ or $[q, p_{23}] \cap [p_{12}, p_{13}] \neq \emptyset$.

In the first case we have that $p_{23} \in F_0 \cap F_1 \cap F_2 \cap F_3$ since we have chosen $p_{23} \in F_0 \cap F_2 \cap F_3$ and $\text{conv}(\{q, p_{12}, p_{13}\}) \subseteq F_1$ as all three of $q, p_{12}, p_{13} \in F_1$ and F_1 is convex.

In the second case we have that the intersection point of $[q, p_{23}]$ and $[p_{12}, p_{13}]$ is in $F_0 \cap F_1 \cap F_2 \cap F_3$. This is because $[q, p_{23}] \subseteq F_2 \cap F_3$ and $[p_{12}, p_{13}] \subseteq F_0 \cap F_1$.

Thus, (3) holds as claimed.

We now define a new collection of sets by setting:

$$\mathcal{F}' := \{F \cap F_0 : F \in \mathcal{F}\}.$$

Since the members of $\mathcal{F}$ are closed and convex and F_0 is compact and convex, each element of $\mathcal{F}'$ is compact and convex. By (2) each set of $\mathcal{F}'$ is nonempty (this is needed since otherwise there cannot be any transversal of $\mathcal{F}'$), and by (3) together with the fact that $\mathcal{F}$ satisfies the $(4,3)$-property, the collection $\mathcal{F}'$ also satisfies the $(4,3)$-property. The theorem now follows from Corollary 6 as every transversal of $\mathcal{F}'$ is also a transversal of $\mathcal{F}$. ■

Acknowledgement

I thank Bart de Keijzer and Branko Grünbaum for helpful discussions.

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