Poisson-Voronoi percolation in the hyperbolic plane with small intensities

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February 15, 2023

Abstract

We consider percolation on the Voronoi tessellation generated by a homogeneous Poisson point process on the hyperbolic plane. We show that the critical probability for the existence of an infinite cluster is asymptotically equal to $\pi\lambda/3$ as $\lambda \to 0$. This answers a question of Benjamini and Schramm [9].

1 Introduction and statement of main result

We will study percolation on the Voronoi tessellation generated by a homogeneous Poisson point process on the hyperbolic plane $\mathbb{H}^2$. That is, with each point of a constant intensity Poisson process on $\mathbb{H}^2$ we associate its Voronoi cell – which is the set of all points of the hyperbolic plane that are closer to it than to any other point of the Poisson process – and we colour each cell black with probability p and white with probability $1-p$, independently of the colours of all other cells. See Figure 1 for a depiction of computer simulations of this process.

We say that *percolation* occurs if there is an infinite connected cluster of black cells. For each intensity $\lambda > 0$ of the underlying Poisson process, the *critical probability* is defined as

$$p_c(\lambda) := \inf\{p : \mathbb{P}_{\lambda,p}(\text{percolation}) > 0\}.$$

To the best of our knowledge, hyperbolic Poisson-Voronoi percolation was first studied by Benjamini and Schramm in the influential paper [9]. Amongst other things, they showed that $0 < p_c(\lambda) < 1/2$ for all $\lambda > 0$; that $p_c(\lambda)$ is a continuous function of λ; and that $p_c(\lambda) \to 0$ as $\lambda \searrow 0$.

They also asked for the asymptotics of $p_c(\lambda)$ as $\lambda \searrow 0$, and specifically for "the derivative at 0". Here we answer this question, by showing:

Theorem 1 *$p_c(\lambda) = (\pi/3) \cdot \lambda + o(\lambda)$ as $\lambda \searrow 0$.*

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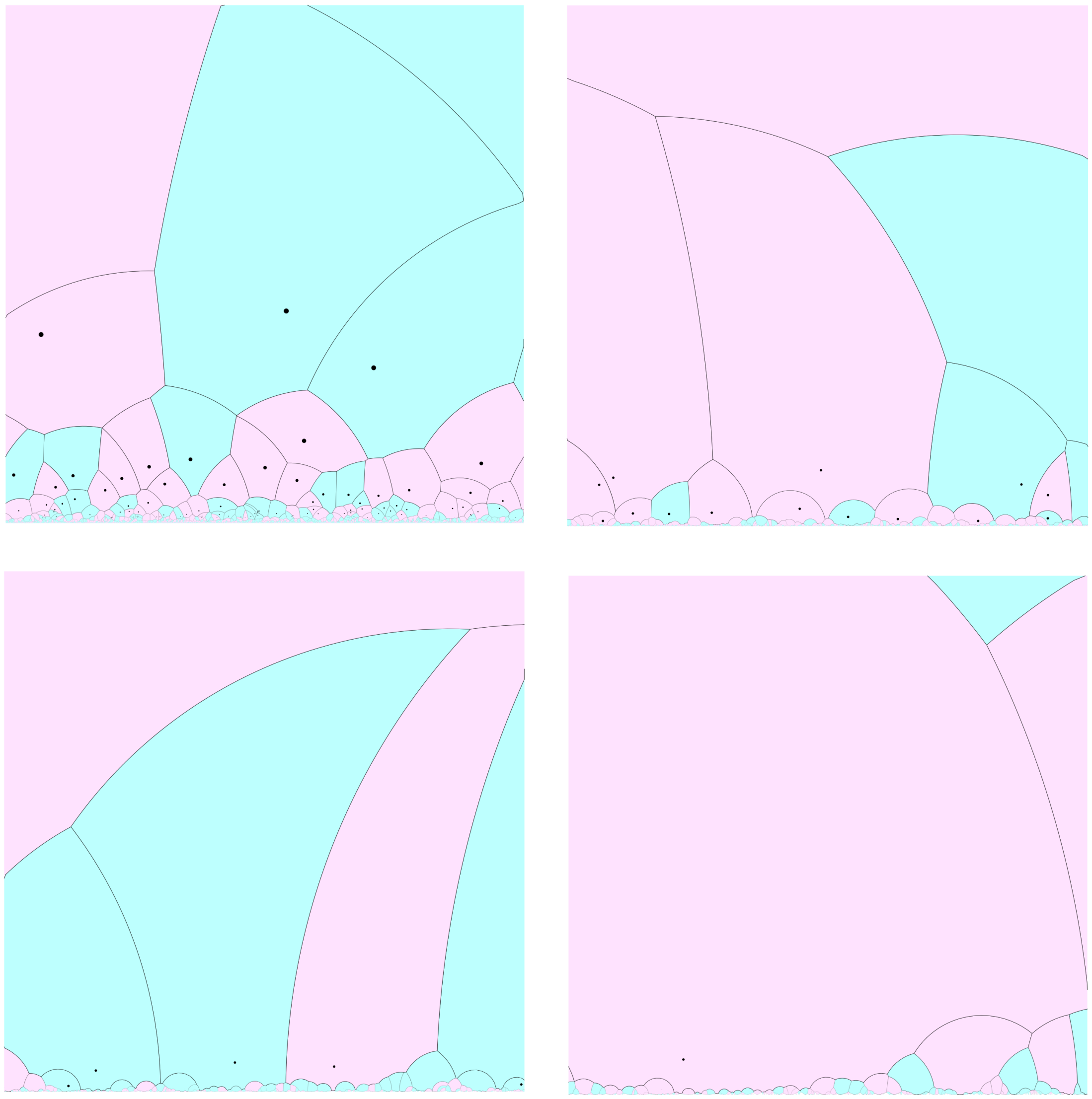

Figure 1: Simulations of hyperbolic Poisson-Voronoi percolation, depicted in the half-plane model. Top left: $\lambda = 1$, top right: $\lambda = \frac{1}{10}$, bottom left: $\lambda = \frac{1}{25}$, bottom right: $\lambda = \frac{1}{50}$; and $p = \frac{1}{2}$ in all cases.

For comparison, Benjamini and Schramm gave an upper bound $p_c(\lambda) \leq \frac{1}{2} - \frac{1}{4\pi\lambda+2}$, whose asymptotics are $\pi\lambda + o(\lambda)$ as $\lambda \searrow 0$.

The results of Benjamini and Schramm highlight striking differences between Poisson-Voronoi percolation in the hyperbolic plane and Poisson-Voronoi percolation in the ordinary, Euclidean plane. For starters, in the latter case it is known [50, 10] that the critical probability equals 1/2 for all values of the intensity parameter λ. More strikingly perhaps is the difference in the behaviour of the number of infinite, black clusters. In the Euclidean case there are no infinite black clusters when $p \leq 1/2$ and precisely one infinite black cluster otherwise (almost surely). For the hyperbolic case, Benjamini and Schramm showed that if $p \leq p_c(\lambda)$ all black clusters are finite; if $p \geq 1 - p_c(\lambda)$ then there is precisely one infinite black cluster; but if

$p_c(\lambda) < p < 1 - p_c(\lambda)$ then there are infinitely many, distinct, infinite, black clusters (almost surely).

Related work. Percolation theory is an active area of modern probability theory with a considerable history, going back to the work of Broadbent and Hammersley [12] in the late fifties. By now there is an immense amount of research articles, mostly centered on percolation on lattices. We direct the reader to the monographs [11, 22] for an overview.

Poisson-Voronoi tessellations (mostly in d-dimensional, Euclidean space) are one of the central models studied in stochastic geometry. They are studied in connection with many different applications and have a long history going back at least to the work of Meijering [34] in the early fifties. For an overview, see the monographs [37, 39] and the references therein.

In the early nineties, Vahidi-Asl and Wierman [46] introduced first passage percolation (a notion related to, but distinct from percolation as we have defined it above) on planar, Euclidean Poisson-Voronoi tessellations. A few years later, Zvavitch [50] proved that in the Euclidean plane almost surely all black clusters are finite when $p \leq 1/2$ (and $\lambda > 0$ arbitrary). About a decade after that Bollobás and Riordan were able to complement this result by showing that, almost surely, there exists an infinite black cluster when $p > 1/2$ (and $\lambda > 0$ arbitrary) – thus establishing $p_c = 1/2$ for planar, Euclidean Poisson-Voronoi percolation. Since then planar, Euclidean Poisson-Voronoi percolation, especially "at criticality", has received a fair amount of additional attention. See e.g. [1, 2, 41, 49]. Poisson-Voronoi percolation on the projective plane was studied by Freedman [17], and Poisson-Voronoi percolation on more general two and three dimensional manifolds was studied by Benjamini and Schramm in [8]. Poisson-Voronoi percolation on higher dimensional Euclidean space has been studied as well, in [4, 3, 15].

Fuchsian groups can be seen as the analogue in the hyperbolic plane of lattices in the Euclidean plane. Lalley [29, 30] studied percolation on Fuchsian groups and amongst other things established that the critical probabilities for "existence" and "uniqueness" of infinite clusters are distinct. Works on continuum percolation models over Poisson processes in the hyperbolic plane include [5, 16, 42, 44, 45]. Aspects of hyperbolic Poisson-Voronoi tessellations besides percolation that have been studied include the (expected) combinatorial structure of their cells, random walks on them and anchored expansions – see for example [6, 7, 13, 19, 26, 27]. Percolation on hyperbolic Poisson-Voronoi tessellations was first studied specifically by Benjamini and Schramm in [9]. In the recent paper [24] the current authors showed that $p_c(\lambda) \to 1/2$ as $\lambda \to \infty$ for Poisson-Voronoi percolation on the hyperbolic plane, proving a conjecture from [9].

Comparing Theorem 1 with Isokawa's formula (stated as Theorem 6 below), we see that our result can be rephrased as : as $\lambda \searrow 0$, the critical probability is asymptotic to the reciprocal of the "typical degree" (defined precisely in Section 2.3 below). A similar phenomenon happens in several Euclidean percolation models when one sends the dimension to infinity. The most well-known result in this direction is probably that for both site and bond percolation on $\mathbb{Z}^d$, we have that $p_c = (1+o_d(1))\cdot(2d)^{-1}$, as was shown concurrently and independently by Gordon [21], Hara and Slade [25] and Kesten [28]. Prior to that there were non-rigorous derivations of the result in the physics literature and Cox and Durret [14] had proved the analogous result for oriented percolation (which is technically easier to analyze). Penrose [36] proved an analogous result for the Gilbert model on d-dimensional Euclidean space as the dimension tends to infinity, and Meester, Penrose and Sarkar [33] extended this result to the random connection model. The analogous phenomenon was shown by Penrose [35] for spread out percolation in fixed dimension when the connections get more and more spread out.

Sketch of the main ideas used in the proof. The intuition guiding the proof is that when λ is small and p is of the same order as λ then black clusters are "locally tree like", in the sense that while there will be some short cycles in the black subgraph of the Delaunay graph, but these will be "rare". (The Delaunay graph is the abstract combinatorial graph whose vertices are the Poisson points and whose edges are precisely those pairs of points whose Voronoi cells meet.) This is also the intuition behind the results on high-dimensional and spread-out percolation mentioned above. In fact, in several of the works cited it is in fact shown that if we scale p as a constant μ divided by the degree (so that the origin has μ black neighbours in expectation) then the cluster of the origin behaves more and more like a Galton-Watson tree with a Poisson(μ) offspring distribution as the dimension grows.

Before going further, it is instructive to informally discuss in a bit more detail the situation for the high-dimensional Gilbert model analyzed by Penrose in [36]. In that model, we build a random graph on a constant intensity Poisson process on $\mathbb{R}^d$ by joining any pair of Poisson points at distance < 1 by an edge. We seek the *critical intensity* λ_c such that there is a.s. no percolation for $\lambda < \lambda_c$ and there is a positive probability of percolation when $\lambda > \lambda_c$. We consider the scenario where d is large and the intensity is $\lambda = \mu \cdot \pi_d^{-1}$ with π_d the volume of the d-dimensional unit ball and $\mu > 0$ a fixed constant. We add the origin o to the Poisson process (note that this a.s. does not change whether or not there is an infinite component), and think of "exploring" the cluster of the origin. We do this in an iterative fashion : we first add the neighbours of the origin to the cluster, then we consider each of these neighbours in turn and add their neighbours to the cluster, and so on. Of course the neighbours of the origin are precisely those Poisson points that fall inside the d-dimensional unit ball B. In particular their number follows a Poisson distribution with mean μ. Once we have already added some points to the cluster of the origin and we consider the neighbours of a previously added point u, we add those Poisson points that fall inside the ball of radius one around u from which we have removed the union of all radius one balls around points we have processed already. Because of the high-dimensional geometry, the volume of this set difference is typically very close to the volume of a unit radius ball with nothing removed – at least during the initial stages of the exploration. In other words, at least during the first few exploration steps, the number of new points that gets added in each exploration step is approximately distributed like a Poisson random variable with mean μ. This naturally leads to the aforementioned connection with Galton-Watson trees with Poisson(μ) offspring distribution. Essentially, the geometric reason why this works out is the "concentration of measure" for high dimensional balls (see for instance [32], Chapter 13) : in high-dimensional Euclidean space, the volume of the unit ball is concentrated near its boundary. From this one can derive that, in a sense that can be made precise, most pairs of points connected by an edge in the Gilbert graph will have distance close to one. Moreover, the mass of a d-dimensional ball is also concentrated near its equator, from which one can derive that – in a sense that can be made precise – for most pairs of points with a common neighbour, their distance will be very close to $\sqrt{2}$ and, more generally, most pairs with graph distance k will have Euclidean distance close to $\sqrt{k}$ (for k fixed).

A similar phenomenon, but maybe even more extreme in a sense, happens in the hyperbolic plane for disks of large radius r : The area of a hyperbolic disk B of large radius r is concentrated near its boundary; and if we take two points at random from B then typically their distance is close to $2r$, the maximal possible distance between any two points in B.

In the current paper, we consider a Poisson point process $\mathcal{Z}$ on the hyperbolic plane with small intensity $\lambda > 0$. We again include the origin and try to analyze the black cluster of the Voronoi cell of the origin in the Voronoi tessellation for $\mathcal{Z} \cup \{o\}$, where the cell of the origin is coloured black and all other cells are each coloured black with probability p and white with probability $1 - p$. By Isokawa's formula (stated as Theorem 6 below) the expected number of cells that are adjacent to the cell of the origin is asymptotic to $\frac{3}{\pi\lambda}$ as λ tends to zero. Moreover, as can be seen either by looking more carefully at Isokawa's computations or by reading some of our arguments below, it can be shown that most neighbours of the origin are at distance close to $r := 2\ln(1/\lambda)$. Note that r goes to infinity as $\lambda \searrow 0$. Now suppose we take $p = \mu \cdot (\pi/3) \cdot \lambda$ with μ a constant and λ small, so that the expected number of black cells neighbouring the cell of the origin is close to μ. If we follow an exploration process analogous to the one described above for the Euclidean Gilbert model, the black cluster will have the property that most pairs of points at graph distance k have distance close to $2kr$ in the hyperbolic metric.

For the upper bound, we will show that if $p = (1 + \varepsilon) \cdot (\pi/3) \cdot \lambda$ and λ is sufficiently small (and $\varepsilon > 0$ is a fixed constant) then the size of the black cluster of the origin stochastically dominates the size of a supercritical Galton-Watson tree. We will use an exploration procedure similar to what we described above for the high-dimensional Gilbert model. During the exploration, when we consider some point z that has already been added to the tree, we make sure to only add those black neighbours of z whose distance to z is within some large constant of r. We also make sure to select a subset $\{z_1, \ldots, z_k\}$ of these neighbours with the property that all angles $\angle z_i z z_j$ are larger than some small constant, and moreover for each z_i there is a ball of radius some large (but fixed) constant that contains no other Poisson points. Essentially because of the geometric phenomena described earlier, it will turn out that in this version of the exploration procedure, for sufficiently small λ, the subgraph of the cluster of the origin we obtain follows the distribution of a certain supercritical Galton-Watson process *exactly*. In contrast, in the high-dimensional Gilbert model the correspondence between the exploration process and a supercritical Galton-Watson process will eventually break down, after a (large) number of exploration steps that depends on the dimension, so that additional techniques and arguments were needed by Penrose [36].

For hyperbolic Poisson-Voronoi percolation the lower bound is technically more involved than the upper bound. This is probably the most novel contribution in our paper, and we believe it might inspire similar arguments applicable to other problems in percolation, notably for high-dimensional models. For percolation on $\mathbb{Z}^d$ and the Gilbert model, a trivial (and "sharp" up to lower order corrections when $d \to \infty$) lower bound is given by a comparison to branching processes : in the former model, the cluster of the origin is stochastically dominated by a Galton-Watson distribution with mean offspring $p \cdot (2d - 1)$ and in the latter with mean offspring $\mu \cdot \pi_d$. So the critical probability satisfies $p_c \geq 1/(2d - 1)$, respectively the critical intensity satisfies $\lambda_c \geq 1/\pi_d$. (For percolation on $\mathbb{Z}^d$ this was in fact already observed by Broadbent and Hammersley [12] and for the Gilbert model by Gilbert [18].) In our case a similar argument does not seem feasible. In the Gilbert model, imagine we have explored part of the cluster of the origin and in doing so have revealed the status of the Poisson process in some region. We now wish to add those neighbours that are not yet part of the cluster of a point that we have previously added to in the cluster. The number of new points added is stochastically dominated by the number of new points we add at the very start of the exploration, when add the neighbours of the origin. In the Poisson-Voronoi percolation model there is no obvious monotonicity of this kind. Once we have revealed the status of the

Poisson point process in some region this can make new edges both more and less likely. As a side remark, let us mention that using the methods in our paper it ought to be possible to show that when $p = (1 - \varepsilon) \cdot (\pi/3) \cdot \lambda$ then, as $\lambda \to 0$, the size of black cluster of the origin will converge in distribution to the size of a Galton-Watson tree with Poisson$(1 - \varepsilon)$ offspring distribution. We do not pursue this here however as it does not appear useful for bounding $p_c(\lambda)$: it will not exclude the possibility that – for any fixed, small $\lambda > 0$ – there is an extremely small, but positive probability that the origin is in an infinite component.

A naive approach that one might try is to compute the expected number of black paths of length k starting at the origin (for λ small and $p = (1 - \varepsilon) \cdot (\pi/3) \cdot \lambda$), and hope to show that this expectation tends to zero as $k \to \infty$. Unfortunately this approach does not seem feasible either. Long paths might revisit the same area many times which introduces dependencies that are difficult to deal with.

In order to circumvent this issue, we introduce what we call *linked sequences of chunks*. As we wish to show percolation does not occur, it suffices to show no infinite black cluster exists with a more generous notion of adjacency, in the form of what we call *pseudo-edges*, that makes the analysis simpler. With each pseudo-edge we associate a "certificate", being the region of the hyperbolic plane that needs to be examined to verify it is indeed a pseudo-edge. We'll say a pseudo-edge on a path P is *good* if it has length close to r and does not make a small angle with the previous pseudo-edge. Otherwise it is *bad*. A linked sequence of chunks is a sequence of paths $P_1, \dots, P_n$ such that: **1)** on each path, every pseudo-edge except the last is good and the last is bad, and **2)** for each $i \geq 2$, either the first point of P_i is close to the certificate of some pseudo-edge of P_{i-1} or the certificate of the first pseudo-edge of P_i intersects one of the certificates of P_{i-1}, and for every other pseudo-edge of P_i its certificate is disjoint from all certificates of all pseudo-edges on $P_1, \dots, P_{i-1}$. We'll first show (Proposition 31 below) that if the cluster of the origin is infinite, then either there exists an infinite path all of whose edges are good, or there exists an infinite linked sequence of chunks starting from the origin. Unlike paths in general, it is technically feasible to give decent bounds on the expected number of good paths, respectively the number of linked sequences of chunks, of some given length n. We are able to obtain bounds that tend to zero with n, which implies no percolation occurs a.s.

Structure of the paper. In the next section, we collect some notations, definitions, facts and tools that we will need in our proofs. Section 3.1 contains some preparatory geometric observations needed in later sections. Section 3.2 contains the proof that $p_c(\lambda) \leq (1 + \varepsilon) \cdot (\pi/3) \cdot \lambda$ for small enough λ (and $\varepsilon > 0$ fixed), and Section 3.3 contains the proof that $p_c(\lambda) \geq (1 - \varepsilon) \cdot (\pi/3) \cdot \lambda$ for small enough λ. We end the paper by suggesting some directions for further work in Section 4.

2 Notation and preliminaries

2.1 Ingredients from hyperbolic geometry

The hyperbolic plane $\mathbb{H}^2$ is a two dimensional surface with constant Gaussian curvature -1. There are many models, i.e. coordinate charts, for $\mathbb{H}^2$ including the Poincaré disk model, the Poincaré half-plane model, and the Klein disk model. A gentle introduction to Gaussian curvature, hyperbolic geometry and these representations of the hyperbolic plane can be found in [38].

Even though we used the Poincaré half-plane model for the visualizations in Figure 1, from now on we will exclusively use the Poincaré disk model. (A computer simulation of Poisson-Voronoi percolation depicted in the Poincaré disk model can be found on the second page of our earlier paper [24].) We briefly recollect its definition and some of the main facts we shall be using in our arguments below, and refer the reader to [38] for proofs and more information.

The Poincaré disk model is constructed by equipping the open unit disk $\mathbb{D} \subseteq \mathbb{R}^2$ with an appropriate metric and area functional. For points $u, v \in \mathbb{D}$, the hyperbolic distance can be given explicitly by

$$\mathrm{dist}_{\mathbb{H}^2}(u,v) = 2\,\mathrm{arcsinh}\left(\frac{\|u-v\|}{\sqrt{(1-\|u\|^2)(1-\|v\|^2)}}\right)$$

where $\|\cdot\|$ denotes the Euclidean norm. Straightforward calculations show that in particular, for $z \in \mathbb{D}$, the Euclidean and hyperbolic distance to the origin are related via:

$$\|z\| = \tanh\left(\mathrm{dist}_{\mathbb{H}^2}(o,z)/2\right). \tag{1}$$

We will use the notations

$$B_{\mathbb{H}^2}(p,r) := \{u \in \mathbb{D} : \mathrm{dist}_{\mathbb{H}^2}(p,u) < r\}, \quad B_{\mathbb{R}^2}(p,r) := \{u \in \mathbb{R}^2 : \|p-u\| < r\},$$

for hyperbolic, respectively Euclidean, disks. A standard fact that we will rely on in the sequel is:

Fact 2 *Every hyperbolic disk is also a Euclidean disk; and every Euclidean disk contained in the open unit disk $\mathbb{D}$ is also a hyperbolic disk.*

(But, the centre and radius of a disk with respect to the hyperbolic metric do not coincide with the centre and radius with respect to the Euclidean metric.)

For any measurable subset $A \subseteq \mathbb{D}$, its *hyperbolic area* is given by

$$\mathrm{area}_{\mathbb{H}^2}(A) := \int_A f(z)\,\mathrm{d}\,z,$$

where

$$f(u) := \frac{4}{(1-\|u\|^2)^2}. \tag{2}$$

From the formulas for hyperbolic distance and hyperbolic area one can derive that:

$$\mathrm{area}_{\mathbb{H}^2}\left(B_{\mathbb{H}^2}(p,r)\right) = 2\pi \cdot (\cosh r - 1). \tag{3}$$

The *hyperbolic polar coordinates* (α, ρ) corresponding to a point $z \in \mathbb{D}$ are $\rho := \mathrm{dist}_{\mathbb{H}^2}(o,z)$ and $\alpha \in [0, 2\pi)$ is the (counterclockwise) angle between the positive x-axis and the line segment oz. Put differently, $z \in \mathbb{D}$ and $\rho \in [0,\infty)$ and $\alpha \in [0,2\pi)$ are such that

$$z = \left(\tanh(\rho/2)\cdot\cos\alpha, \tanh(\rho/2)\cdot\sin\alpha\right). \tag{4}$$

In several computations in the paper we'll use the *change of variables to hyperbolic polar coordinates*. By this we of course just mean applying the substitution (4). We will always apply it to integrals of the form $\int_{\mathbb{D}} g(z)f(z)\,\mathrm{d}\,z$ with f as given by (2) above, to obtain:

$$\int_{\mathbb{D}} g(z)f(z)\,\mathrm{d}\,z = \int_0^\infty \int_0^{2\pi} g\left(\tanh(\rho/2)\cdot\cos\alpha, \tanh(\rho/2)\cdot\sin\alpha\right)\sinh\rho\,\mathrm{d}\,\alpha\,\mathrm{d}\,\rho. \tag{5}$$

A *hyperbolic geodesic* or *hyperbolic line segment* between two points $a, b \in \mathbb{D}$ is the shortest path between a and b with respect to the hyperbolic metric. If there is a (Euclidean) line passing through a, b and the origin o, then the hyperbolic geodesic between a and b is just the (ordinary) line segment between them. Otherwise, the hyperbolic geodesic between a, b can be constructed geometrically as follows. We let $C \subseteq \mathbb{R}^2$ be the unique (Euclidean) circle with $a, b \in C$ and such that it hits the boundary $\partial\mathbb{D}$ of the unit disk at right angles. The points a, b divide $C \setminus \{a, b\}$ into two circle segments. The hyperbolic geodesic between a and b is the one contained in $\mathbb{D}$.

A *hyperbolic line* $\ell \subseteq \mathbb{D}$ is either the intersection of an Euclidean line through the origin with the open unit disk $\mathbb{D}$, or the intersection of $\mathbb{D}$ with a circle C hitting $\partial\mathbb{D}$ at right angles.

If $a, b, c \in \mathbb{D}$ then we use $\angle abc$ to denote the angle the hyperbolic line segment between a and b and the hyperbolic line segment between b and c make in the common point b. (In general, when $a, b, c, \in \mathbb{D}$ do not lie on a hyperbolic line, there will be two possible interpretations of this angle, one in $(0, \pi)$ and one in $(\pi, 2\pi)$. As is usual, we shall always take the smaller of the two.) A *hyperbolic triangle* is the set of the three hyperbolic line segments defined by three (non-collinear) points $a, b, c \in \mathbb{D}$. A critical tool is *hyperbolic law of cosines*. A proof of this result can for instance be found in [43] (page 81).

Lemma 3 *For a hyperbolic triangle, with sides of length a, b, c and respective opposite angles α, β, γ (see Figure 2):*

$$\cosh(c) = \cosh(a)\cosh(b) - \sinh(a)\sinh(b)\cos(\gamma).$$

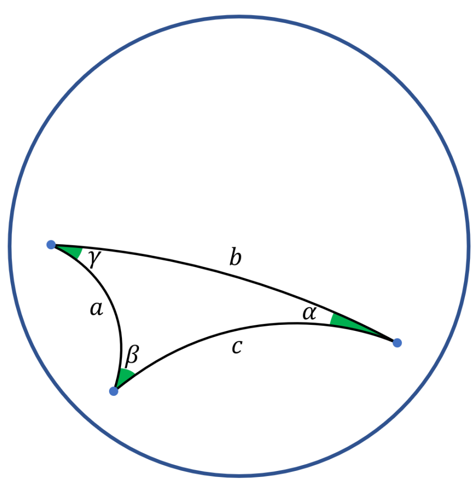

Figure 2: A hyperbolic triangle with sides of length a, b, and c and respective opposite angles α, β, and γ. The blue circle is the boundary of the Poincaré disk.

A hyperbolic ray is defined analogously to a ray in Euclidean geometry. That is, if ℓ is a hyperbolic line then any $p \in \ell$ divides $\ell \setminus \{p\}$ into two connected components. Each of these is a *ray emanating from* p. For distinct $p, s \in \mathbb{D}$ and $\vartheta \in (0, \pi)$ we let $\text{sect}(p, s, \vartheta)$ denote the set of all hyperbolic rays emanating from p that make an angle of no more than ϑ with the ray emanating from p through s. In particular, in the Poincaré disk model, if $p = o$ is the origin then $\text{sect}(p, s, \vartheta)$ looks like a (Euclidean) disk sector of opening angle 2ϑ with bisector the ray emanating from p through s. See Figure 3. For any other p, the set $\text{sect}(p, s, \vartheta)$ can be obtained by applying a suitable hyperbolic isometry to such a disk sector.

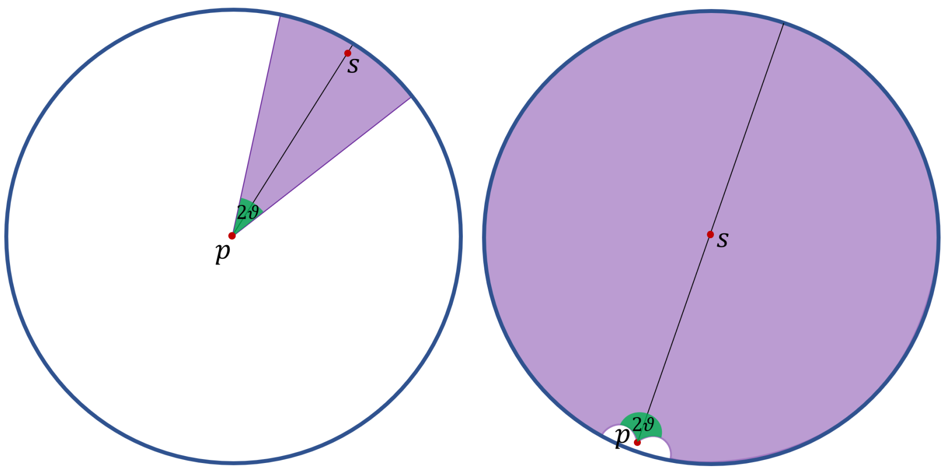

Figure 3: The black segment is the ray emanating from p through s and the purple region is the set $\text{sect}(p, s, \vartheta)$. On the left, $p = o$. On the right, the sector after applying an isometry that maps p away from the origin and s to the origin. The blue circle is the boundary of the Poincaré disk.

A *hyperbolic isometry* is a bijection $\varphi : \mathbb{D} \to \mathbb{D}$ that preserves hyperbolic distance, i.e. $\text{dist}_{\mathbb{H}^2}(u, v) = \text{dist}_{\mathbb{H}^2}(\varphi(u), \varphi(v))$ for all $u, v \in \mathbb{D}$. For any two points in $x, y \in \mathbb{D}$, there exists a unique hyperbolic line ℓ through x and y, and there exists a hyperbolic isometry φ such that $\varphi[\ell]$ is an open interval given by the line segment between $(-1, 0)$ and $(1, 0)$, $\varphi(x) = o$ is the origin and $\varphi(y)$ is on the positive x-axis. In addition to distance, hyperbolic isometries preserve angles, in the sense that if $c_1, c_2 \subseteq \mathbb{D}$ are curves (e.g. hyperbolic line segments, circles, ...) that meet in the point p at angle α, then the curves $\varphi[c_1], \varphi[c_2]$ meet in the point $\varphi(p)$ at the same angle α. Hyperbolic isometries also preserve hyperbolic area. That is,

$$\text{area}_{\mathbb{H}^2}(\varphi[A]) = \text{area}_{\mathbb{H}^2}(A), \tag{6}$$

for all (measurable) $A \subseteq \mathbb{D}$ and every hyperbolic isometry φ. What is more, if $\varphi : \mathbb{D} \to \mathbb{D}$ is a hyperbolic isometry then, for any (integrable) $g : \mathbb{D} \to \mathbb{R}$ we have

$$\int_{\mathbb{D}} g(z) f(z) \,\mathrm{d}\, z = \int_{\mathbb{D}} g(\varphi(u)) f(u) \,\mathrm{d}\, u, \tag{7}$$

with f as defined by (2). (An easy way to see this is to first note it follows trivially from (6) when g is the indicator function of some measurable $A \subseteq \mathbb{D}$. From this it easily follows for step-functions $g = \sum_{i=1}^n a_i 1_{A_i}$. Then it follows for an arbitrary measurable function, by approximating it arbitrarily closely by step functions.)

2.2 Hyperbolic Poisson point processes

In the rest of this paper $\mathcal{Z}$ will denote a homogeneous Poisson point process (PPP) on the hyperbolic plane. Analogously to homogeneous Poisson point processes on the ordinary, Euclidean plane, a homogeneous Poisson process $\mathcal{Z}$ of intensity λ on the hyperbolic plane is characterized completely by the properties that **a)** for each (measurable) set $A \subseteq \mathbb{D}$ the random variable $|\mathcal{Z} \cap A|$ is Poisson distributed with mean $\lambda \cdot \text{area}_{\mathbb{H}^2}(A)$, and **b)** if $A_1, \ldots, A_m \subseteq \mathbb{D}$ are (measurable and) disjoint then the random variables $|\mathcal{Z} \cap A_1|, \ldots, |\mathcal{Z} \cap A_m|$ are independent. In the light of the formula for $\text{area}_{\mathbb{H}^2}(.)$ above, we can alternatively view $\mathcal{Z}$ as an *inhomogeneous* Poisson point process on the ordinary, Euclidean plane $\mathbb{R}^2$ with intensity function

$$u \mapsto \lambda \cdot 1_{\mathbb{D}}(u) \cdot f(u),$$

with f given by (2) above.

Throughout the remainder, we attach to each point of $\mathcal{Z}$ a randomly and independently chosen colour. (Black with probability p and white with probability $1-p$.) We let $\mathcal{Z}_{\text{b}}$ denote the black points and $\mathcal{Z}_{\text{w}}$ the white points of $\mathcal{Z}$. In the language of for instance [31], we can view $\mathcal{Z}$ as a *marked* Poisson point process, the marks corresponding to the colours.

We will rely heavily on a specific case of the Slivniak-Mecke formula, which is our weapon of choice for counting tuples of points $z_1, \ldots, z_k \in \mathcal{Z}$ satisfying a given property. Before stating it, we remind the reader that formally speaking a Poisson process on $\mathbb{R}^2$ is a random variable that takes values in the space Ω_{PPP} of locally finite subsets of $\mathbb{R}^2$, equipped with the sigma algebra generated by the family events of the form : a given Borel B set contains precisely k points.

Theorem 4 (Slivniak-Mecke formula) *Let $\mathcal{Z}$ be a homogeneous hyperbolic Poisson point process of intensity λ, and let Let $g : \mathbb{D}^k \times \Omega_{PPP} \to [0, \infty)$ be a nonnegative, measurable function. Then*

$$\mathbb{E}\left[\sum_{\substack{z_1,\ldots,z_k \in \mathcal{Z} \\ \textit{distinct}}} g(z_1, ..., z_k, \mathcal{Z})\right]$$
$$=$$
$$\lambda^k \int_{\mathbb{D}} \cdots \int_{\mathbb{D}} \mathbb{E}\left[g(x_1, ..., x_k, \mathcal{Z} \cup \{x_1, ..., x_k\})\right] f(x_1) \ldots f(x_k) \, \mathrm{d}\, x_k \ldots \mathrm{d}\, x_1,$$

with f given by (2).

As mentioned above, the version we state here is a specific case of a more general result. The general version can for instance be found in [37], as Corollary 3.2.3. (The version we present here is the special case of Corollary 3.2.3 in [37] when the ambient space $E = \mathbb{R}^2$ and the intensity measure has density $\lambda \cdot 1_{\mathbb{D}} \cdot f$ with f as in (2).) The Slivniak-Mecke formula is sometimes also called Mecke formula or Campbell-Mecke formula in the literature.

We shall be applying the following consequence of the Slivniak-Mecke formula, that is tailored to our situation where $\mathcal{Z} = \mathcal{Z}_{\mathrm{b}} \cup \mathcal{Z}_{\mathrm{w}}$ and membership in $\mathcal{Z}_{\mathrm{b}}$ is determined via independent, p-biased coin flips.

Corollary 5 *For $\lambda > 0$ and $0 \leq p \leq 1$, let $\mathcal{Z} = \mathcal{Z}_b \cup \mathcal{Z}_w$ be as above and let $g : \mathbb{D}^k \times \Omega_{PPP} \to [0, \infty)$ be a nonnegative, measurable function. Then*

$$\mathbb{E}\left[\sum_{\substack{z_1,...,z_k \in \mathcal{Z}_b \\ \text{distinct}}} g(z_1, ..., z_k, \mathcal{Z})\right]$$
$$=$$
$$(p\lambda)^k \int_{\mathbb{D}} \cdots \int_{\mathbb{D}} \mathbb{E}\left[g(u_1, ..., u_k, \mathcal{Z} \cup \{u_1, ..., u_k\})\right] f(u_1) \ldots f(u_k) \, \mathrm{d}\, u_1 \ldots \mathrm{d}\, u_k,$$

with f given by (2)*.*

Proof. Let us write

$$S := \sum_{\substack{z_1,...,z_k \in \mathcal{Z} \\ \text{distinct}}} g(z_1, ..., z_k, \mathcal{Z}), \quad S_{\mathrm{b}} := \sum_{\substack{z_1,...,z_k \in \mathcal{Z}_{\mathrm{b}} \\ \text{distinct}}} g(z_1, ..., z_k, \mathcal{Z}).$$

We imagine first "revealing" the locations of the Poisson points $\mathcal{Z}$, but not yet the colours/coin flips. For any fixed, locally finite $\mathcal{U} \subseteq \mathbb{R}^d$ we have

$$\mathbb{E}\left(S | \mathcal{Z} = \mathcal{U}\right) = \sum_{\substack{u_1,...,u_k \in \mathcal{U}, \\ \text{distinct}}} f(u_1, \ldots, u_k, \mathcal{U}),$$

while

$$\begin{aligned} \mathbb{E}\left(S_{\mathrm{b}} | \mathcal{Z} = \mathcal{U}\right) &= \sum_{\substack{u_1,...,u_k \in \mathcal{U}, \\ \text{distinct}}} f(u_1, \ldots, u_k, \mathcal{U}) \cdot \mathbb{P}(u_1, \ldots, u_k \text{ are coloured black}) \\ &= p^k \cdot \left(\sum_{\substack{u_1,...,u_k \in \mathcal{U}, \\ \text{distinct}}} f(u_1, \ldots, u_k, \mathcal{U}) \right) = p^k \cdot \mathbb{E}\left(S | \mathcal{Z} = \mathcal{U}\right). \end{aligned}$$

This holds for every locally finite $\mathcal{U}$, which implies

$$\mathbb{E} S_{\mathrm{b}} = p^k \cdot \mathbb{E} S.$$

The result now follows by applying the Slivniak-Mecke formula to $\mathbb{E}S$. ■

2.3 Hyperbolic Poisson-Voronoi tessellations

For $\mathcal{U} \subseteq \mathbb{D}$ a countable point set and $u \in \mathcal{U}$ we will denote the corresponding *hyperbolic Voronoi cell* of u by:

$$C(u; \mathcal{U}) := \{v \in \mathbb{D} : \mathrm{dist}_{\mathbb{H}^2}(u, v) \leq \mathrm{dist}_{\mathbb{H}^2}(u', v) \text{ for all } u' \in \mathcal{U}\}.$$

We will usually suppress the second argument, and just write $C(u)$ for the Voronoi cell of u; and $\mathcal{U}$ will usually be either a homogeneous hyperbolic Poisson process $\mathcal{Z}$ or $\mathcal{Z} \cup \{o\}$, such a Poisson process with the origin added in.

The *hyperbolic Delaunay graph* for $\mathcal{U}$ is the abstract combinatorial graph with vertex set $\mathcal{U}$ and an edge uu' if the Voronoi cells $C(u), C(u')$ meet. So we can alternatively view hyperbolic Poisson-Voronoi percolation as site-percolation on the hyperbolic Poisson-Delaunay graph. We say that $u, u' \in \mathcal{U}$ are *adjacent* if they share an edge in the Poisson-Delaunay graph (in other words, the Voronoi cells $C(u), C(u')$ have at least one point in common). In this case we will also say that u, u' are *neighbours*.

We point out that if $v \in C(u) \cap C(u')$ then it must hold that $\operatorname{dist}_{\mathbb{H}^2}(v, u) = \operatorname{dist}_{\mathbb{H}^2}(v, u')$ and $B_{\mathbb{H}^2}(v, \operatorname{dist}_{\mathbb{H}^2}(v, u)) \cap \mathcal{U} = \emptyset$. In other words u, u' are adjacent if and only if there is hyperbolic disk B such that $u, u' \in \partial B$ and $B \cap \mathcal{U} = \emptyset$. Although we shall not be using this in the present paper, it may be instructive to the reader to point out the following. By this last observation, Fact 2 and some relatively straightforward probabilistic considerations : if $\mathcal{Z}$ is a homogeneous, hyperbolic Poisson process then, almost surely, the Voronoi tessellations for $\mathcal{Z}$ with respect to the ordinary, Euclidean metric and the Voronoi tessellation with respect to the hyperbolic metric as defined above have the same combinatorial structure, in the sense that $z, z' \in \mathcal{Z}$ are adjacent in the Euclidean tessellation if and only if they are adjacent in the hyperbolic tessellation. See Lemma 5 in our previous paper [24].

In many of our arguments below we will be considering the Voronoi tessellation for $\mathcal{Z} \cup \{o\}$, a homogeneous, hyperbolic Poisson process with the origin added in. We refer to the origin as the *typical point* and to its Voronoi cell $C(o)$ as the *typical cell*. See Figure 4 for computer simulations of the typical cell, shown in the Poincaré disk model, for various choices of λ.

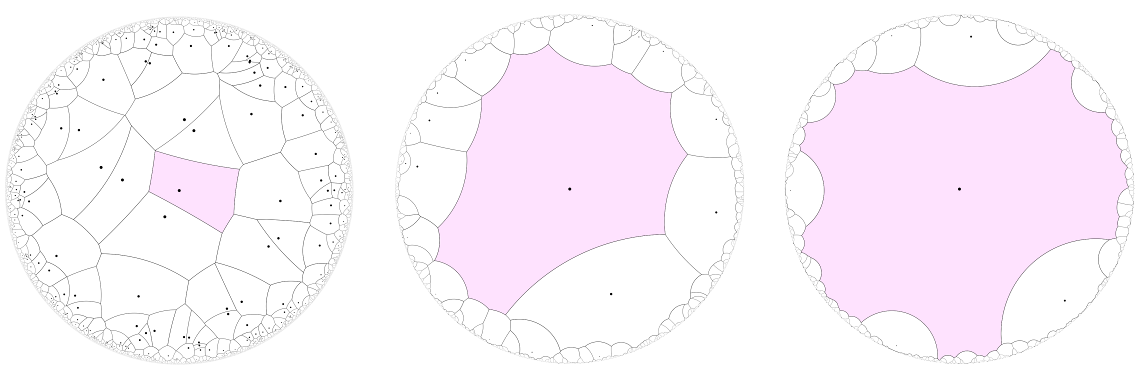

Figure 4: Computer simulations of the typical cell (highlighted), shown in the Poincaré disk model. Left: $\lambda = 1$, middle : $\lambda = \frac{1}{10}$, right: $\lambda = \frac{1}{50}$.

We define the *typical degree* D as

$$D := |\{z \in \mathcal{Z} : o, z \text{ are neighbours in the Voronoi tessellation for } \mathcal{Z} \cup \{o\}\}| . \tag{8}$$

The typical degree for homogeneous, hyperbolic Poisson processes has previously been studied by Isokawa [27]. She gave the following exact formula for its expectation, valid for all values of the intensity parameter $\lambda > 0$.

Theorem 6 (Isokawa's formula) *If D is as given by* (8) *then*

$$\mathbb{E}D = 6 + \frac{3}{\pi\lambda}.$$

What is important for us is the asymptotics $\mathbb{E}D \sim \frac{3}{\pi\lambda}$ as $\lambda \searrow 0$. An independent, alternative derivation of these asymptotics, using conceptually simple arguments that are similar to the ones we will use in the present paper, is given in Chapter 3 of the doctoral thesis of the first author [23].

It may be instructive for the reader (but does not enter into the arguments in the present paper) to point out that as $\lambda \to \infty$ the expected typical degree tends to 6, the value for the expected typical degree in planar, Euclidean Poisson-Voronoi tessellations [34].

It may also be instructive to have another look at the simulations in Figure 4 (and 1), which indeed suggest as λ decreases the typical cell tends to have more adjacencies.

3 Proofs

3.1 Drawing trees in the hyperbolic plane

We will find it useful to consider graphs embedded in the hyperbolic plane. From now on every graph T in the rest of the paper, has vertices $V(T) \subseteq \mathbb{D}$ that are points in the Poincaré disk, and we identify each edge $uv \in E(T)$ with the (hyperbolic) geodesic line segment between u and v.

Definition 7 *For $\rho, w, \vartheta > 0$ we say that T is a (ρ, w, ϑ)-*tree *if it is connected and*

(i) *For every edge $uv \in E(T)$ we have $\rho - w < \text{dist}_{\mathbb{H}^2}(u, v) < \rho + w$;*

(ii) *If the edges $uv, uw \in E(T)$ have a common endpoint u, then the angle $\angle vuw$ they make at u is at least ϑ.*

We will speak of a (ρ, w, ϑ)-*path* if the (ρ, w, ϑ)-tree T is a path.

The following proposition verifies that we are justified in using the word "tree" : a (ρ, w, ϑ)-tree is also a tree in the graph theoretical sense, at least when the ρ parameter is sufficiently large.

Proposition 8 *For every $w, \vartheta > 0$ there exists $\rho_0 = \rho_0(w, \vartheta)$ such that, for all $\rho \geq \rho_0$, every (ρ, w, ϑ)-tree is acyclic.*

We have to postpone the proof until some necessary preparations are out of the way. If T is a (ρ, w, ϑ)-tree then clearly the hyperbolic distance $\text{dist}_{\mathbb{H}^2}(u, v)$ between any two vertices $u, v \in V(T)$ is upper bounded by $\text{dist}_T(u, v) \cdot (\rho + w)$. Here and in the rest of the paper, dist_T denotes the *graph distance*, i.e. the number of edges on the (unique) u, v-path in T.

The following proposition shows that for ρ sufficiently large this trivial upper bound is close to being tight.

Proposition 9 *For every $w, \vartheta > 0$ there exists $\rho_0 = \rho_0(w, \vartheta)$ and $K = K(w, \vartheta)$ such that, for all $\rho \geq \rho_0$, every (ρ, w, ϑ)-tree and for every two vertices $u, v \in V(T)$:*

$$\text{dist}_{\mathbb{H}^2}(u, v) \geq \text{dist}_T(u, v) \cdot (\rho - K).$$

Again we postpone the proof until we have made some additional observations.

If G is a graph and $uv \in E(G)$ then we denote by $G_{u\setminus v}$ the connected component of $G\setminus\{v\}$ that contains u. In other words, $G_{u\setminus v}$ is the subgraph induced on all nodes other than v that can be reached from u using a path that avoids v.

The following observation is an important ingredient in the proof of our main result.

Proposition 10 *For all $w, \vartheta_1, \vartheta_2 > 0$ there exist $\rho_0 = \rho_0(w, \vartheta_1, \vartheta_2)$ and $h = h(w, \vartheta_1, \vartheta_2)$ such that, for all $\rho \geq \rho_0$, every (ρ, w, ϑ_1)-tree T and every edge $uv \in E(T)$:*

$$\bigcup_{x \in V(T_{u\setminus v})} B_{\mathbb{H}^2}(x, \rho + w) \subseteq B_{\mathbb{H}^2}(v, h) \cup \operatorname{sect}(v, u, \vartheta_2).$$

Again we have to postpone the proof until we have made some more preparatory observations. The next few Lemma's in this section are technical (hyperbolic) geometric considerations that are intermediate steps in the proof of the three propositions above. When reading the paper for the first time, the reader may wish to only read the lemma statements and skip over the proofs.

We start with a relatively straightforward, but for us useful, consequence of the hyperbolic cosine rule.

Lemma 11 *For every $\gamma_0 > 0$, there exists $K = K(\gamma_0)$ such that the following holds for every hyperbolic triangle. Let the lengths of the sides be a, b, and c and the respective opposing angles α, β, γ as in Figure 2. If $\gamma \geq \gamma_0$ then*

$$c \geq a + b - K.$$

Proof. We can assume without loss of generality that $\gamma_0 \leq \pi/2$. For any hyperbolic triangle satisfying the hypothesis of the lemma, we have

$$\begin{aligned} e^c &\geq \cosh(c) \\ &= \cosh(a)\cosh(b) - \sinh(a)\sinh(b)\cos(\gamma) \\ &\geq \cosh(a)\cosh(b) - \sinh(a)\sinh(b)\cos(\gamma_0) \\ &\geq \frac{e^{a+b}}{4}(1 - \cos(\gamma_0)). \end{aligned}$$

The second line is the hyperbolic law of cosines (Lemma 3). The third line follows as $0 < \gamma < \pi$ and $\cos(.)$ is decreasing on $[0, \pi)$. The fourth line uses that $\cosh(a)\cosh(b) \geq \frac{e^{a+b}}{4}$ and $\sinh(a)\sinh(b) \leq \frac{e^{a+b}}{4}$ and $\gamma_0 \leq \pi/2$ (by assumption). Taking logs, we find

$$c \geq a + b + \ln\left(\frac{1 - \cos\gamma_0}{4}\right) =: a + b - K.$$

■

Next, we show that if u, v are at least $r + d_0$ apart with d_0 a (large) constant and r arbitrary then, from the point of view of v (i.e. if we isometrically map v to the origin of the Poincaré disk), a ball of large constant radius r around u will be contained in a sector of small opening angle.

Lemma 12 *For every $\vartheta > 0$ there exists a $d_0 = d_0(\vartheta) > 0$ such that, for all $u, v \in \mathbb{H}^2$ and $r > 0$, if $\mathrm{dist}_{\mathbb{H}^2}(u,v) > r + d_0$ then*

$$B_{\mathbb{H}^2}(u,r) \subseteq \mathrm{sect}\,(v,u,\vartheta)\,.$$

(See Figure 5.)

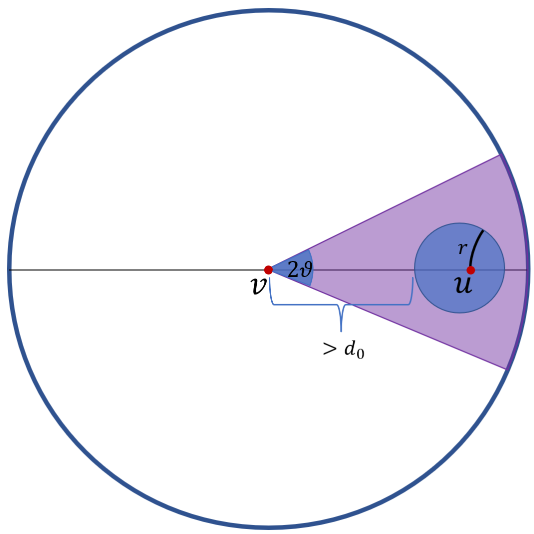

Figure 5: An example of $B_{\mathbb{H}^2}(u,r) \subseteq \mathrm{sect}\,(v,u,\vartheta)$.

Proof. We let d_0 be a large constant, to be determined in the course of the proof. Applying a suitable isometry if needed, we can assume without loss of generality that $v = o$ is the origin and u lies on the positive x-axis. We recall that $B := B_{\mathbb{H}^2}(u,r)$ is also a Euclidean disk. Its Euclidean center must lie on the x-axis. (This is for instance easily seen by noting that the reflection in the x-axis is a hyperbolic isometry that fixes u, and hence leaves B invariant.)

By the triangle inequality and the assumption that $\mathrm{dist}_{\mathbb{H}^2}(u,v) > r + d_0$, we have

$$\mathrm{dist}_{\mathbb{H}^2}(v,z) > d_0 \text{ for all } z \in B.$$

By (1), for each $z \in B$ we have

$$1 > \|z\| > \tanh(d_0/2) \geq 1 - \delta,$$

where δ is a small constant to be determined shortly, and the final inequality holds provided we have chosen d_0 sufficiently large. Since B is contained in the annulus $\mathbb{D} \setminus B_{\mathbb{R}^2}(o, 1-\delta)$, its Euclidean radius is no more than $\delta/2$.

As its Euclidean centre lies on the x-axis, it is clear that $\delta = \delta(\vartheta)$ can be chosen so that $D \subseteq \mathrm{sect}\,(v,u,\vartheta)$. ■

Our next observation is another intermediate step in the proofs of Propositions 8, 9 and 10. It will be used only in the proof of Lemma 14 that immediately succeeds it in the text.

Lemma 13 *Suppose the (Euclidean) circle C intersects the unit circle $\partial\mathbb{D}$ at right angles, and intersects the x-axis at an angle $\vartheta > 0$. If r denotes the (Euclidean) radius of C and u denotes the intersection point of C and the x-axis that falls inside $\mathbb{D}$, then*

$$r = \frac{1 - \|u\|^2}{2\|u\| \sin\vartheta}.$$

Proof. Let c denote the centre of C. Since C hits the unit circle at a right angle, we have $\|c\| = \sqrt{1+r^2}$. In the Euclidean triangle with corners o, u, c, the angle at u equals $\pi/2 + \vartheta$. See Figure 6.

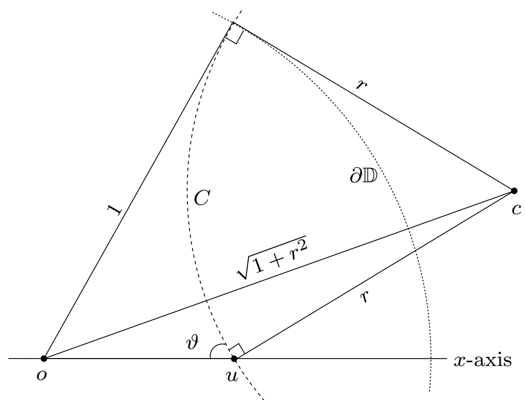

Figure 6: Illustration of the proof of Lemma 13.

Applying the Euclidean cosine rule to the triangle with corners o, u, c we find

$$1 + r^2 = \|u\|^2 + r^2 - 2\|u\| r \cos(\pi/2 + \vartheta) = \|u^2\| + r^2 + 2\|u\| r \sin(\vartheta).$$

The claimed expression follows by reorganizing this last identity. ■

The next lemma makes the following perhaps rather counterintuitive observation. While $\mathbb{D} \setminus \text{sect}\,(u, v, \vartheta)$ looks "large" from the vantage point of u (i.e. if we isometrically map u to the origin of the Poincaré disk, the sector looks only like a small "slice"), provided u, v are sufficiently far apart, from the vantage point of v the set $\mathbb{D} \setminus \text{sect}\,(u, v, \vartheta) \subseteq \text{sect}\,(v, u, \vartheta)$ is actually contained in a small "slice".

Lemma 14 *For all $\vartheta > 0$ there exists $d_0 = d_0(\vartheta) > 0$ such that, for all $u, v \in \mathbb{D}$, if* $\text{dist}_{\mathbb{H}^2}(u, v) \geq d_0$, *then* $\mathbb{D} \setminus \text{sect}\,(u, v, \vartheta) \subseteq \text{sect}\,(v, u, \vartheta)$.

(See Figure 7.)

Proof. Applying a suitable isometry if needed, we can assume without loss of generality that $v = o$ is the origin and u lies on the positive x-axis.

The boundary of $\text{sect}\,(u, v, \vartheta)$ consists of two rays emanating from u that each make an angle ϑ with the hyperbolic line through u and v at u (and a circle segment that is part of the unit circle $\partial\mathbb{D}$). In Euclidean terms, these two rays are circle segments of circles that each meet the x-axis in u at an angle of ϑ, and intersect the boundary of the unit circle $\partial\mathbb{D}$ at

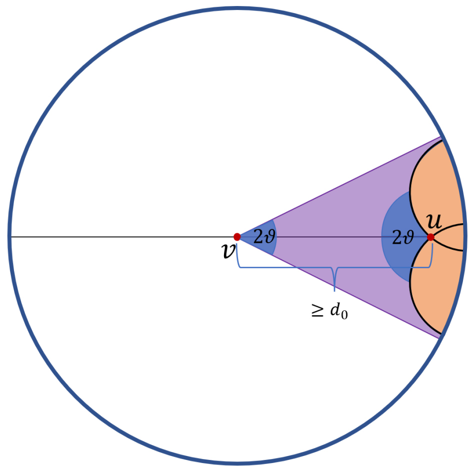

Figure 7: The orange region is $\mathbb{D}\setminus \text{sect}\,(u,v,\vartheta)$. The union of the purple and the orange region is $\text{sect}\,(v,u,\vartheta)$.

right angles. (See Figure 7.) Let us call these circles C_1, C_2. Both circles meet the x-axis in u at angle ϑ. Hence, by Lemma 13, both have Euclidean radius $(1-\|u\|^2)/(2\|u\|\sin\vartheta)$.

We have $\|u\| \geq \tanh(d_0/2)$ which approaches one as d_0 approaches infinity. In particular, for any constant $\delta > 0$, we can choose the constant d_0 so that both C_1, C_2 have Euclidean radius $< \delta$, and both intersect the x-axis at a point within distance δ of $(1,0)$. In other words, we'll have

$$\mathbb{D} \setminus \text{sect}\,(u,v,\vartheta) \subseteq B_{\mathbb{R}^2}((1,0),3\delta) \cap \mathbb{D}.$$

Clearly, if we take $\delta = \delta(\vartheta)$ small enough, it now follows that $\mathbb{D}\setminus\text{sect}\,(u,v,\vartheta) \subseteq \text{sect}\,(u,v,\vartheta)$, as desired. ■

As our final preparation for the proofs of Propositions 8 and 9, we show that if T is a (ρ,w,ϑ)-tree and $uv \in E(T)$ an edge then the vertices of $T_{u\setminus v}$ are all contained in a small sector from the vantage point of v, provided ρ is chosen sufficiently large.

Lemma 15 *For every $w, \vartheta_1, \vartheta_2 > 0$ there exists a $\rho_0 = \rho_0(w,\vartheta_1,\vartheta_2)$ such that, for every $\rho \geq \rho_0$, every (ρ,w,ϑ_1)-tree T and every edge $uv \in E(T)$:*

$$V(T_{u\setminus v}) \subseteq \text{sect}\,(v,u,\vartheta_2)\,.$$

Proof. We can and do assume, without loss of generality, that $\vartheta_1 < \frac{1}{1000}$ and $\vartheta_2 < \vartheta_1/2$; and we set $\rho_0 := w + d_0$ with $d_0 = d_0(\vartheta_2)$ as provided by Lemma 14.

We will use induction on the number of edges $m := |E(T)|$ of T. The statement is clearly true when $m \leq 1$. Let us thus assume that, for some $m \in \mathbb{N}$, the statement holds for all (ρ,w,ϑ_1)-trees with $< m$ edges, let T be an arbitrary (ρ,w,ϑ_1)-tree with m edges, and pick an arbitrary edge $uv \in E(T)$. If u is a leaf then $T_{u\setminus v} = \{u\}$ and the statement is trivial. So we can and do assume u has at least two neighbours. Let us denote the neighbours of u as $N(u) = \{v, x_1, \ldots, x_k\}$ and let T_i denote the connected component of $T \setminus ux_i$ containing x_i. By the induction hypothesis $V(T_i) \subseteq \text{sect}\,(u,x_i,\vartheta_2)$ for $i = 1,\ldots,k$. Since the line segments

uv and ux_i make an angle $\geq \vartheta_1$, it follows that $\operatorname{sect}(u, v, \vartheta_2) \cap \operatorname{sect}(u, x_i, \vartheta_2) = \emptyset$. In other words

$$V(T_{u \setminus v}) \subseteq \{u\} \cup \bigcup_{i=1}^{k} V(T_i) \subseteq \mathbb{D} \setminus \operatorname{sect}(u, v, \vartheta_2) \subseteq \operatorname{sect}(v, u, \vartheta_2),$$

where the last inclusion follows by Lemma 14 and our choice of ρ_0. (Using that $\operatorname{dist}_{\mathbb{H}^2}(v, u) \geq \rho_0 - w \geq d_0$ by choice of ρ_0.) ■

Proof of Proposition 8. We let ρ_0 be a sufficiently large constant, to be determined during the course of the proof. Let T be an arbitrary (ρ, w, ϑ)-tree for some $\rho \geq \rho_0$, and suppose it contains a cycle C. Observe that as C is a connected subgraph of T, it is also a (r, w, ϑ)-tree. Let v_1, v_2, v_3 be three vertices that are consecutive on C. Clearly $C_{v_1 \setminus v_2} = C \setminus v_2$ is just C with v_2 and both incident edges removed. We let $\vartheta' > 0$ be a sufficiently small constant, to be determined more precisely shortly. By the previous lemma,

$$V(C) \setminus \{v_2\} \subseteq \operatorname{sect}\left(v_2, v_1, \vartheta'\right), \tag{9}$$

assuming we chose ρ_0 sufficiently large. By symmetry (considering $C_{v_3 \setminus v_2}$) we also have

$$V(C) \setminus \{v_2\} \subseteq \operatorname{sect}\left(v_2, v_3, \vartheta'\right). \tag{10}$$

Since $\angle v_1 v_2 v_3 > \vartheta$, provided we chose ϑ' sufficiently small, we have that

$$\operatorname{sect}\left(v_2, v_1, \vartheta'\right) \cap \operatorname{sect}\left(v_2, v_3, \vartheta'\right) = \emptyset.$$

Combining this with (9) and (10), would imply that $V(C) = \{v_2\}$, contradicting our assumption that C is a cycle. ■

Proof of Proposition 9. We let ρ_0, K be sufficiently large constants, to be specified more precisely during the course of the proof. We will use induction on $n := \operatorname{dist}_T(u, v)$.

The base case, when $n = 1$ is trivial by definition of (ρ, w, ϑ)-tree – provided we chose $K \geq w$.

Let us then assume the statement is true for $n - 1$ and let $u = v_0, \ldots, v_n = v$ be a (u, v)-path in T. By Proposition 8, having chosen ρ_0 sufficiently large, we know there is precisely one path between any pair of vertices. By the induction hypothesis

$$\operatorname{dist}_{\mathbb{H}^2}(v_1, v_n) \geq (n - 1) \cdot (\rho - K),$$

and by Lemma 15, assuming we chose ρ_0 sufficiently large, we have that

$$v \in V(T_{v_2 \setminus v_1}) \subseteq \operatorname{sect}(v_1, v_2, \vartheta/2).$$

By definition of (ρ, w, ϑ)-tree we have $\angle u v_1 v_2 > \vartheta$. It follows that $\angle u v_1 v > \vartheta/2$. Applying Lemma 11, we find

$$\operatorname{dist}_{\mathbb{H}^2}(u, v) \geq \operatorname{dist}_{\mathbb{H}^2}(u, v_1) + \operatorname{dist}_{\mathbb{H}^2}(v_1, v) - K',$$

for some constant $K' = K'(\vartheta/2)$ provided by Lemma 11. Hence,

$$\operatorname{dist}_{\mathbb{H}^2}(u, v) \geq (\rho - w) + (n - 1) \cdot (\rho - K) - K' \geq n \cdot (\rho - C),$$

assuming (without loss of generality) we have chosen $K > w + K'$. ■

The final ingredient we need for the proof of Proposition 10 is the following observation. It states that if the distance between u and v is between $r-w$ and $r+w$, where w is constant and r arbitrary, then a ball of radius $r+w$ around u is contained in the union of a ball of constant radius h around v and a sector of constant opening angle ϑ

Lemma 16 *For every $w, \vartheta > 0$ there exists $h = h(w, \vartheta)$ such that for all $r > 0$ and all $u, v \in \mathbb{D}$ with $r - w < \mathrm{dist}_{\mathbb{H}^2}(u,v) < r+w$ we have*

$$B_{\mathbb{H}^2}(u, r+w) \subseteq B_{\mathbb{H}^2}(v, h) \cup \mathrm{sect}\,(v, u, \vartheta)\,.$$

Proof. Applying a suitable isometry if needed, we can assume without loss of generality $v = o$ is the origin and that u lies on the positive x-axis.

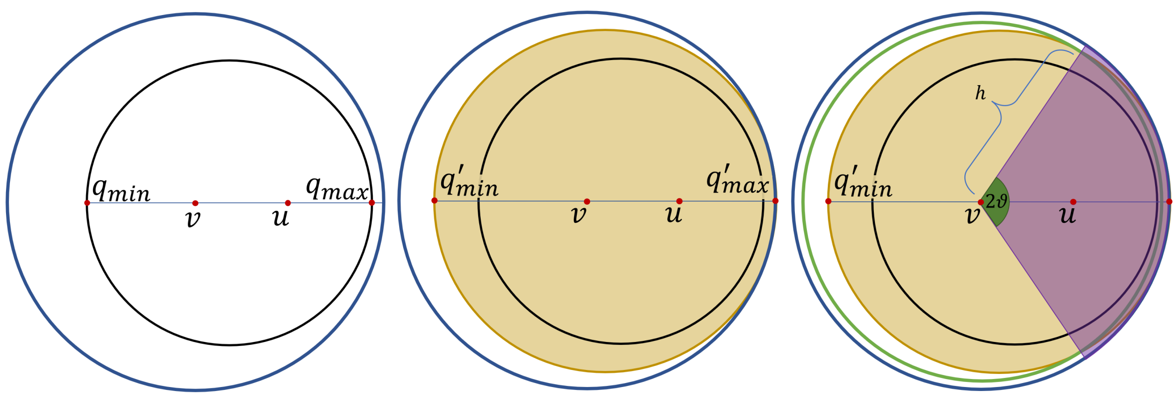

Figure 8: On the right, the disk with the black boundary is $B_{\mathbb{H}^2}(u, r+w)$. In the centre, the gold disk is D. On the left, the purple sector is $\mathrm{sect}\,(v, u, \vartheta)$ and the disk with the green boundary is $B_{\mathbb{H}^2}(v, h)$.

We recall that the hyperbolic disk $B_{\mathbb{H}^2}(u, r+w)$ is also a Euclidean disk $B_{\mathbb{R}^2}(z, t)$. The Euclidean center z must lie on the x-axis (an easy way to see this is that reflection in the x-axis is a $\mathbb{H}^2$-isometry that leaves $B_{\mathbb{H}^2}(u, r+w)$ invariant). Let $q_{\min} = (x_{\min}, 0)$ and $q_{\max} = (x_{\max}, 0)$ be the points where the circle $\partial B_{\mathbb{H}^2}(u, r+w)$ intersects the x-axis. See Figure 8, left. (So the Euclidean center of $B_{\mathbb{H}^2}(u, r+w)$ is the midpoint $(q_{\min} + q_{\max})/2$ between these two points.) Since $v = (0,0) \in B_{\mathbb{H}^2}(u, r+w)$ we have $x_{\min} \leq 0 \leq x_{\max}$. As the points $q_{\min}, v = o, u$ lie on the x-axis, which is a hyperbolic line, we have

$$\mathrm{dist}_{\mathbb{H}^2}(q_{\min}, u) = \mathrm{dist}_{\mathbb{H}^2}(q_{\min}, v) + \mathrm{dist}_{\mathbb{H}^2}(u, v),$$

giving

$$\mathrm{dist}_{\mathbb{H}^2}(q_{\min}, o) = \mathrm{dist}_{\mathbb{H}^2}(q_{\min}, u) - \mathrm{dist}_{\mathbb{H}^2}(u, v) \leq r + w - (r - w) = 2w.$$

By (1) it follows that $\|q_{\min}\| \leq \tanh(w) < 1$. In other words, $-\tanh(w) \leq x_{\min} \leq 0$. We set

$$q'_{\min} := (-\tanh(w), 0), \quad q'_{\max} := (1, 0),$$

and let

$$D := B_{\mathbb{R}^2}\left(\left(\frac{1-\tanh(w)}{2}, 0\right), \frac{1+\tanh(w)}{2}\right),$$

be the Euclidean disk with center $(q'_{\min} + q'_{\max})/2 = ((1 - \tanh(w))/2, 0)$ and radius $(1 + \tanh(w))/2$. Put differently, D is the disk with $q'_{\min}, q'_{\max}$ on its boundary, that meets the x-axis at a right angle at both points. This is not a hyperbolic disk (it is what is called a horocycle), but we do have that $B_{\mathbb{H}^2}(u, r + w) \subseteq D \subseteq \mathbb{D}$. See Figure 8, middle.

Let $R := D \setminus \operatorname{sect}(v, u, \vartheta)$ be the part of D that is not contained in the sector $\operatorname{sect}(v, u, \vartheta)$. Then

$$h := \sup\{\operatorname{dist}_{\mathbb{H}^2}(z, o) : z \in R\} < \infty,$$

since all points of R are at least some positive Euclidean distance away from the boundary of the unit disk. See Figure 8, right. Evidently we have

$$B_{\mathbb{R}^2}(u, r + w) \subseteq D \subseteq R \cup \operatorname{sect}(v, u, \vartheta) \subseteq B_{\mathbb{H}^2}(v, h) \cup \operatorname{sect}(v, u, \vartheta),$$

as desired. (Note that h depends only on w and ϑ.) ■

Proof of Proposition 10. We let ρ_0 and h be large constants, to be determined in the course of the proof. Let $\rho \geq \rho_0$, T be an arbitrary (ρ, w, ϑ_1)-tree, $uv \in E(T)$ an arbitrary edge and $x \in T_{u\setminus v}$ an arbitrary vertex.

We first assume that $x \neq u$. Then $\operatorname{dist}_T(u, x) \geq 2$ by Proposition 8, assuming without loss of generality we have chosen rho_0 sufficiently large. And, assuming ρ_0 was chosen appropriately, by Proposition 9 we have

$$\operatorname{dist}_{\mathbb{H}^2}(v, x) \geq 2(\rho - K),$$

where K is as provided by the proposition.

By Lemma 15, provided we chose ρ_0 appropriately, we have

$$x \in \operatorname{sect}(v, u, \vartheta_2/2).$$

Applying Lemma 12, provided we chose ρ_0 sufficiently large, we have

$$B_{\mathbb{H}^2}(x, r + w) \subseteq \operatorname{sect}(v, x, \vartheta_2/2) \subseteq \operatorname{sect}(v, u, \vartheta_2).$$

Let us thus consider the situation when $x = u$. In this case we can apply Lemma 16 to show that, provided we chose ρ_0 and h appropriately large,

$$B_{\mathbb{H}^2}(u, r + w) \subseteq B_{\mathbb{H}^2}(v, h) \cup \operatorname{sect}(v, u, \vartheta_2).$$

This concludes the proof. ■

3.2 The upper bound

Here we will show the following proposition, which constitutes half of our main result.

Proposition 17 *For every $\varepsilon > 0$, there exists a $\lambda_0 = \lambda_0(\varepsilon) > 0$ such that for all $0 < \lambda < \lambda_0$, we have $p_c(\lambda) \leq (1+\varepsilon) \cdot (\pi/3) \cdot \lambda$.*

In order to prove this, we add the origin o to the Poisson point process $\mathcal{Z}$ and colour it black, and consider the black component of o in the Delaunay graph for $\mathcal{Z} \cup \{o\}$. It suffices to show that, when $p \geq (1+\varepsilon) \cdot (\pi/3) \cdot \lambda$ and λ is sufficiently small, with positive probability the origin will be in an infinite black component. This is because all edges not involving the origin are also in the Delaunay graph for $\mathcal{Z}$, and the origin a.s. has only finitely many neighbours (by Isokawa's formula), so if the origin is in an infinite component then removing the origin may split its component into several components but at least one of these will be infinite.

For $u, v \in \mathbb{D}$ and $r, w, \vartheta > 0$ we define the region

$$C(u, v, r, w, \vartheta) := \{x \in \mathbb{D} : r - w < \operatorname{dist}_{\mathbb{H}^2}(u, x) < r + w \text{ and } \angle vux > \vartheta\}.$$

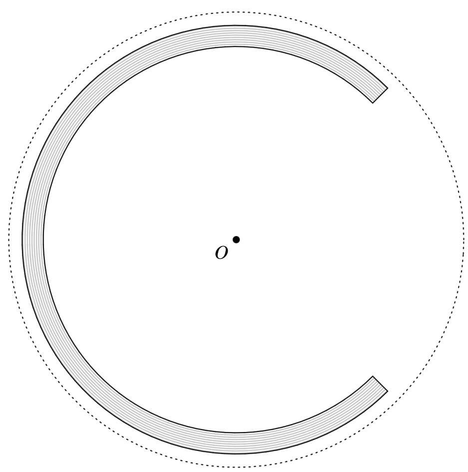

Figure 9: The C-shaped region $C(o, (1/2, 0), 3, 1/2, \pi/4)$.

We choose to use the notation $C(., ., ., .)$ because the region is shaped like the letter C, at least for some choices of the parameters. See Figure 9.

Given $u \neq v \in \mathbb{D}$ and $r, w, \vartheta, h > 0$ we define $\mathcal{X} = \mathcal{X}(u, v, r, w, \vartheta, h)$ and $X = X(u, v, r, w, \vartheta, h)$ by:

$$\mathcal{X} := \left\{ z \in \mathcal{Z}_b : \begin{array}{l} z \in C(u, v, r, w, \vartheta) \text{ and;} \\ \angle z'uz > \vartheta \text{ for all } z' \neq z \in \mathcal{Z}_b \cap C(u, v, r, w, \vartheta), \text{ and;} \\ \mathcal{Z} \cap B_{\mathbb{H}^2}(z, h) = \{z\}, \text{ and;} \\ \exists \text{ a disk } B \text{ such that } u, z \in \partial B, B \cap \mathcal{Z} = \emptyset, \operatorname{diam}_{\mathbb{H}^2}(B) < r + w. \end{array} \right\},$$

$$X := |\mathcal{X}|. \tag{11}$$

We point out that the probability distribution of X does not depend on the choice of u and v (as long as they are distinct – otherwise the definition does not make sense).

Proposition 18 *For every $w, \vartheta > 0$ there exist $h = h(w, \vartheta)$ and $r_0 = r_0(w, \vartheta)$ such that, for all $\lambda > 0, 0 < p < 1$ and $r > r_0$, the size of the black cluster of o stochastically dominates the size of a Galton-Watson branching process with offspring distribution X.*

Proof. We can assume without loss of generality $\vartheta < \pi$ (otherwise $X = 0$ by definition, and the theorem holds trivially). We set $\vartheta' \ll \vartheta$ be a small constant to be determined during the course of the proof, let $h = h(w, \vartheta, \vartheta')$ be as provided by Proposition 10, and we let r_0 be a large constant, to be determined more precisely during the course of the proof.

We consider an "exploration process" that iteratively constructs a (r, w, ϑ)-tree T rooted at the origin. Recall that r will be chosen larger than r_0. At every iteration there will be a (finite) number of nodes, some of which are *explored* and some *unexplored*. The node v having been explored means that all children of v have already been added to the tree.

For each iteration $i \geq 1$, let T_i denote the tree we have constructed at the end of iteration i. We let $\mathcal{E}_i$ denote the set of nodes of T_i that have been explored at the end of iteration i and $\mathcal{U}_i$ be the set of nodes in T_i that have not yet been explored; and we set $\mathcal{E}_0 = \emptyset, \mathcal{U}_0 = \{o\}$.

If in any iteration there are no unexplored nodes, i.e. $\mathcal{U}_i = \emptyset$, then the construction of the tree stops and the final result is $T = T_i$. Otherwise, if in each iteration we always have at least one unexplored node, then we continue indefinitely in which case of course $T := \bigcup_i T_i$.

In the first iteration we add the set $\mathcal{X}_1 := \mathcal{X}(o, (\tanh(r/2), 0), r, w, \vartheta, h)$ defined above to the tree, as the children of the origin. To be more precise we define T_1 by $V(T_1) = \{o\} \cup \mathcal{X}_1, E(T_1) = \{oz : z \in \mathcal{X}_1\}$. We set $\mathcal{E}_1 = \{o\}, \mathcal{U}_1 = \mathcal{X}_1$. In particular, the number of children of the origin will be $X_1 \stackrel{d}{=} X$.

In any subsequent iteration i, assuming $\mathcal{U}_{i-1} \neq \emptyset$ (otherwise the construction process will have finished), we pick an arbitrary unexplored node $u_i \in \mathcal{U}_{i-1}$, and we denote by v_i its parent in T_{i-1}. The children of u_i will be $\mathcal{X}_i := \mathcal{X}(u_i, v_i, r, w, \vartheta, h)$, and we let $X_i := |\mathcal{X}_i|$ denote the number of children of u_i. We update by defining T_i via $V(T_i) = V(T_{i-1}) \cup \mathcal{X}_i, E(T_i) = E(T_{i-1}) \cup \{u_i z : z \in \mathcal{X}_i\}$, and setting $\mathcal{E}_i = \mathcal{E}_{i-1} \cup \{u_i\}, \mathcal{U}_i = (\mathcal{U}_{i-1} \setminus \{u_i\}) \cup \mathcal{X}_i$.

We point out that, since T_i clearly is a (r, w, ϑ)-tree, it follows from Proposition 8 that we do not include any point twice (put differently that $\mathcal{X}_i \cap \mathcal{X}_j = \emptyset$ for all $i \neq j$), provided r_0 was chosen sufficiently large.

It remains to see that the size of the tree T constructed via this process dominates the size of a Galton-Watson tree with offspring distribution X. In order to do this, we will first establish that $X_1, X_2, \ldots$ are independent, and then that $X_2, X_3, \ldots$ are i.i.d. and finally that X_2 stochastically dominates $X = X_1$.

We will consider, for each iteration i, a (random) region R_i that will be *revealed* by the exploration process during the i-th iteration. Here we mean by "reveal" that the exploration process will use information about $\mathcal{Z} \cap R_i$ in order to determine $\mathcal{X}_i$, but – crucially – after the i-th iteration the exploration process will not have uncovered any information on the status of the Poisson point process $\mathcal{Z}$ outside of $R_1 \cup \cdots \cup R_i$.

We define $\mathcal{X}_i^+ \supseteq \mathcal{X}_i$ by

$$\mathcal{X}_i^+ := \left\{ z \in \mathcal{Z}_b : \begin{array}{l} z \in C(u_i, v_i, r, w, \vartheta) \text{ and;} \\ \angle z' u z > \vartheta \text{ for all } z' \neq z \in \mathcal{Z}_b \cap C(u_i, v_i, r, w, \vartheta). \end{array} \right\},$$

setting $u_1 = o, v_1 := (\tanh(r/2), 0)$ so that the definition also applies for $i = 1$. We let T_i^+ be the tree on vertex set $\{o\} \cup \mathcal{X}_1^+ \cup \cdots \cup \mathcal{X}_i^+$, rooted at $u_1 = o$ and where the children of u_j are $\mathcal{X}_j^+$ for each $j = 1, \ldots, i$. Clearly T_i^+ is also a (r, w, ϑ)-tree for each i.

Given $\mathcal{X}_1^+, \ldots, \mathcal{X}_{i-1}^+$ and u_i, we can determine $\mathcal{X}_i^+$ by revealing the status of the Poisson process inside $C(u_i, v_i, r, w, \vartheta)$. In order to now determine $\mathcal{X}_i \subseteq \mathcal{X}_i^+$ we need to determine, for each $z \in \mathcal{X}_i^+$ whether $B_{\mathbb{H}^2}(z,h) \cap \mathcal{Z} = \{z\}$ and there exists a disk B such that $u_i, z \in \partial B$, $\mathcal{Z} \cap B = \emptyset$ and $\mathrm{diam}_{\mathbb{H}^2}(B) < r + w$. Any such B is contained in $B_{\mathbb{H}^2}(u_i, r+w) \cap B_{\mathbb{H}^2}(z, r+w)$. Hence, given $\mathcal{X}_1^+, \ldots, \mathcal{X}_{i-1}^+$ and u_i, we can determine $\mathcal{X}_i$ by revealing the status of the Poisson process inside the region

$$R_i := \bigcup_{z \in C(u_i, v_i, r, w, \vartheta)} (B_{\mathbb{H}^2}(u_i, r+w) \cap B_{\mathbb{H}^2}(z, r+w)) \cup \bigcup_{z \in \mathcal{X}_i^+} B_{\mathbb{H}^2}(z, h).$$

By Proposition 10 (applied to the tree consisting of a single edge $u_i z$) and the choice of h, r_0, we have

$$B_{\mathbb{H}^2}(z, r+w) \subseteq B_{\mathbb{H}^2}(u_i, h) \cup \mathrm{sect}\left(u_i, z, \vartheta'\right),$$

for all $z \in C(u_i, v_i, r, w, \vartheta)$. It follows that

$$\begin{aligned} R_i &\subseteq B_{\mathbb{H}^2}(u_i, h) \cup \left(\bigcup_{z \in C(u_i, v_i, r, w, \vartheta)} \mathrm{sect}\left(u_i, z, \vartheta'\right) \right) \\ &\subseteq B_{\mathbb{H}^2}(u_i, h) \cup \left(\mathbb{D} \setminus \mathrm{sect}\left(u_i, v_i, \vartheta - \vartheta'\right)\right) \\ &=: S_i, \end{aligned} \tag{12}$$

(We point out that the arguments giving (12) also apply to the case when $i = 1$, showing that $\mathcal{X}_1$ is determined completely by $\mathcal{Z} \cap S_1$.)

On the other hand, we also have

$$R_j \subseteq B_{\mathbb{H}^2}(u_j, r+w) \cup \bigcup_{z \in \mathcal{X}_j^+} B_{\mathbb{H}^2}(z, h),$$

for every j. Hence

$$\begin{aligned} R_1 \cup \cdots \cup R_{i-1} &\subseteq \left(\bigcup_{j=1}^{i-1} B_{\mathbb{H}^2}(u_i, r+w) \right) \cup \left(\bigcup_{v \in \{o\} \cup \mathcal{X}_1^+ \cup \cdots \cup \mathcal{X}_{i-1}^+} B_{\mathbb{H}^2}(v, h) \right) \\ &\subseteq B_{\mathbb{H}^2}(u_i, h) \cup \left(\bigcup_{\substack{v \in \{o\} \cup \mathcal{X}_1^+ \cup \cdots \cup \mathcal{X}_{i-1}^+, \\ v \neq u_i}} B_{\mathbb{H}^2}(v, r+w) \right) \\ &= B_{\mathbb{H}^2}(u_i, h) \cup \left(\bigcup_{v \in V((T_{i-1}^+)_{v_i \setminus u_i})} B_{\mathbb{H}^2}(v, r+w) \right) \\ &\subseteq B_{\mathbb{H}^2}(u_i, h) \cup \mathrm{sect}\left(u_i, v_i, \vartheta'\right), \end{aligned} \tag{13}$$

where we apply Proposition 10 in the last line, and we assume we chose r_0 sufficiently large.

Combining (12) and (13), having chosen ϑ' sufficiently small, we see that

$$(R_1 \cup \cdots \cup R_{i-1}) \cap S_i \subseteq B_{\mathbb{H}^2}(u_i, h).$$

Moreover, for $i > 1$, by construction of the exploration process we have

$$\mathcal{Z} \cap B_{\mathbb{H}^2}(u_i, h) = \{u_i\}.$$

Hence, for $i > 1$, given $\mathcal{X}_1, \mathcal{X}_1^+, \ldots, \mathcal{X}_{i-1}, \mathcal{X}_{i-1}^+, u_1, \ldots, u_i$, the random set $\mathcal{X}_i$ is completely determined by $\mathcal{Z} \cap S_i'$ where

$$S_i' := S_i \setminus B_{\mathbb{H}^2}(u_i, h) = \mathbb{D} \setminus \left(B_{\mathbb{H}^2}(u_i, h) \cup \operatorname{sect}\left(u_i, v_i, \vartheta - \vartheta'\right)\right).$$

(For $i = 1$ it might be the case that $B_{\mathbb{H}^2}(o, h)$ contains point of $\mathcal{Z}$. So we cannot say that $\mathcal{X}_1$ is completely determined by $\mathcal{Z} \cap S_1'$. It is however completely determined by $\mathcal{Z} \cap S_1$.)

We consider an isometry $\varphi : \mathbb{D} \to \mathbb{D}$ satisfying $\varphi(u_i) = o$ and that $\varphi(v_i)$ lies on the positive x-axis. Thus

$$\varphi\left[S_i'\right] = \mathbb{D} \setminus \left(B_{\mathbb{H}^2}(o, h) \cup \operatorname{sect}\left(o, v_1, \vartheta - \vartheta'\right)\right) = S_1',$$

and of course $\varphi\left[B_{\mathbb{H}^2}(u_i, h)\right] = B_{\mathbb{H}^2}(o, h)$.

Since S_i' does not not intersect the areas $R_1, \ldots, R_{i-1}$ revealed by previous iterations and it is isometric to S_1' for each $i > 1$, we find that $X_1, X_2, \ldots$ are independent, and for $i > 1$ in fact

$$X_i \stackrel{d}{=} \left(X_1 \middle| |\mathcal{Z} \cap B_{\mathbb{H}^2}(o, h)| = 0\right) \qquad \text{(for } i > 1.)$$

(We write $|B_{\mathbb{H}^2}(o, h) \cap \mathcal{Z}| = 0$ and not $B_{\mathbb{H}^2}(o, h) \cap \mathcal{Z} = \{o\}$ since $o \notin \mathcal{Z}$.)

If $X_1, X_2, \ldots$ were i.i.d. then T would be a Galton-Watson tree with offspring distribution X_1. In our case, we can describe the situation by saying the tree T consists of a root attached to X_1 independent copies of a Galton-Watson tree with offspring distribution $\tilde{X}_1 = (X_1 | \mathcal{Z} \cap B_{\mathbb{H}^2}(o, h) = \emptyset)$. We also point out that the sequence $X_1, X_2, \ldots$ completely determines the size of the tree T, via

$$|V(T)| = \inf\{n : X_1 + \cdots + X_n \leq n - 1\}.$$

(See for instance Sections 1.5 and 1.6 of [20]. Note that while the discussion there focuses on $X_1, X_2, \ldots$ i.i.d., the above equation holds much more generally. See the remark following Definition 1.14 in [20].)

To conclude the proof, we note that, provided we chose r_0 sufficiently large, $B_{\mathbb{H}^2}(o, h) \cap C(u_1, v_1, r, w, \vartheta) = \emptyset$. Therefore, any point of $\mathcal{Z}$ in $B_{\mathbb{H}^2}(o, h)$ can only "prevent" the formation of edges between o and some $z \in \mathcal{Z}_b \cap C(u_1, v_1, r, w, \vartheta)$. In particular, for any $k, \ell \in \mathbb{N} \cup \{0\}$ we have

$$\mathbb{P}\left(X_1 \geq k \middle| |\mathcal{Z} \cap B_{\mathbb{H}^2}(o, h)| = \ell\right) \leq \mathbb{P}\left(X_1 \geq k \middle| |\mathcal{Z} \cap B_{\mathbb{H}^2}(o, h)| = 0\right) = \mathbb{P}(\tilde{X}_1 \geq k),$$

which gives

$$\mathbb{P}(X_1 \geq k) = \sum_{\ell=0}^{\infty} \mathbb{P}\left(X_1 \geq k \middle| |\mathcal{Z} \cap B_{\mathbb{H}^2}(o,h)| = \ell\right) \mathbb{P}\left(|\mathcal{Z} \cap B_{\mathbb{H}^2}(o,h)| = \ell\right) \leq \mathbb{P}(\tilde{X}_1 \geq k).$$

In other words, $\tilde{X}_1$ stochastically dominates X_1. By Strassen's theorem ([40]; an elementary proof of the version we need can for instance be found in Section 2.3 of [47]) there is a coupling of the sequence $X_1, X_2, X_3, \dots$ and an i.i.d. sequence $Y_1, Y_2, \dots$ such that $Y_i \stackrel{d}{=} X_1$ for all i and $X_i \geq Y_i$ almost surely. The sequence $Y_1, Y_2, \dots$ can be used to generate a Galton-Watson tree T' with offspring distribution X_1. We have $|V(T')| = \inf\{n \geq 1 : Y_1 + \dots + Y_n \leq n-1\}$. Hence, (under the coupling, almost surely) $|V(T')| \leq |V(T)|$. In particular the size of the black cluster of the origin (which is at least $|V(T)|$) stochastically dominates $|V(T')|$. This is what needed to be shown. ■

Having established Proposition 18, in order to prove Proposition 17 it suffices to show that, for λ sufficiently small and $p = (1+\varepsilon) \cdot (\pi/3) \cdot \lambda$, there is a choice of w, ϑ, h, r such that ($h = h(w,\vartheta), r \geq r_0(w,\vartheta)$ with $h(.,.), r_0(.,.)$ as specified in Proposition 18, and) $\mathbb{E}X > 1$. More specifically, we'll keep w, ϑ (and h) constant, but we'll let r depend on λ.

We break the argument down in a series of relatively straightforward lemmas.

Lemma 19 *For every $w, \lambda > 0$ and all $p \leq 10\lambda$ the following holds, setting $r := 2\log(1/\lambda)$. Writing*

$$X_{\mathbf{I}} := |\mathcal{Z}_b \cap B_{\mathbb{H}^2}(o, r-w)|,$$

we have

$$\mathbb{E}X_{\mathbf{I}} \leq 1000e^{-w}.$$

Proof. If $r - w < 0$ then clearly $X_{\mathbf{I}} = 0$ almost and we are done. So we can assume $r - w \geq 0$. Clearly

$$\begin{aligned} \mathbb{E}X_{\mathbf{I}} &= p\lambda \cdot \text{area}_{\mathbb{H}^2}\left(B_{\mathbb{H}^2}(o, r-w)\right) \\ &= p\lambda \cdot 2\pi\left(\cosh(r-w) - 1\right) \\ &\leq 1000\lambda^2 e^{r-w} \\ &= 1000e^{-w}, \end{aligned}$$

using that $\cosh(x) - 1 \leq e^x$ for $x \geq 0$, and the choice of r. ■

Lemma 20 *For every $w, \lambda, \vartheta > 0$ and all $p \leq 10\lambda$ the following holds, setting $r := 2\log(1/\lambda)$ and letting $v \in \mathbb{D} \setminus \{o\}$ be an arbitrary (fixed) point. Writing*

$$X_{\mathbf{II}} := |\mathcal{Z}_b \cap B_{\mathbb{H}^2}(o, r+w) \cap \text{sect}\,(o, v, \vartheta)|,$$

we have

$$\mathbb{E}X_{\mathbf{II}} \leq 1000\vartheta e^{w}.$$

Proof. Clearly

$$\begin{array}{rcl} \mathbb{E}X_{\mathbf{II}} & = & p\lambda \, \text{area}_{\mathbb{H}^2}\left(B_{\mathbb{H}^2}(o,r+w)\cap \text{sect}\,(o,v,\vartheta)\right) \\ & = & p\lambda \cdot \frac{2\vartheta}{2\pi} \cdot 2\pi\,(\cosh(r+w)-1) \\ & \leq & 1000\vartheta\lambda^2 e^{r+w} \\ & = & 1000\vartheta e^{w}, \end{array}$$

as before using that $\cosh(x)-1 \leq e^x$ and the choice of r. ■

Lemma 21 *For every $w, \lambda, \vartheta > 0$ and all $p \leq 10\lambda$ the following holds, setting $r := 2\log(1/\lambda)$. Writing*

$$X_{\mathbf{III}} := \left| \left\{ (z_1,z_2) \in \mathcal{Z}_b \times \mathcal{Z}_b : \begin{array}{l} z_1, z_2 \in B_{\mathbb{H}^2}(o,r+w), \text{ and;} \\ z_1 \neq z_2, \text{ and;,} \\ \angle z_1 o z_2 < \vartheta. \end{array} \right\} \right|,$$

we have

$$\mathbb{E}X_{\mathbf{III}} \leq 1000\vartheta e^{2w}.$$

Proof. By Corollary 5:

$$\mathbb{E}X_{\mathbf{III}} = p^2\lambda^2 \int_{\mathbb{D}}\int_{\mathbb{D}} \mathbb{E}\left[g(z_1,z_2,\mathcal{Z}\cup\{z_1,z_2\})\right] f(z_1)f(z_2)\,\mathrm{d}\,z_2\,\mathrm{d}\,z_2,$$

where f is as given by (2) and

$$g(u_1,u_2,\mathcal{U}) = 1_{\left\{ \begin{array}{l} u_1, u_2 \in B_{\mathbb{H}^2}(o,r+w), \text{ and;} \\ u_1 \neq u_2, \text{ and;,} \\ \angle u_1 o u_2 < \vartheta. \end{array} \right\}}.$$

In other words

$$\begin{array}{rcl} \mathbb{E}X_{\mathbf{III}} & = & p^2\lambda^2 \displaystyle\int_{B_{\mathbb{H}^2}(o,r+w)} \int_{B_{\mathbb{H}^2}(o,r+w)\cap\text{sect}(o,z_1,\vartheta)} f(z_1)f(z_2)\,\mathrm{d}\,z_2\,\mathrm{d}\,z_1 \\ & = & p^2\lambda^2 \displaystyle\int_{B_{\mathbb{H}^2}(o,r+w)} \text{area}_{\mathbb{H}^2}\left(B_{\mathbb{H}^2}(o,r+w)\cap\text{sect}\,(o,z_1,\vartheta)\right) f(z_1)\,\mathrm{d}\,z_1 \\ & = & p^2\lambda^2 \displaystyle\int_{B_{\mathbb{H}^2}(o,r+w)} \left(\frac{2\vartheta}{2\pi}\right) \cdot \text{area}_{\mathbb{H}^2}\left(B_{\mathbb{H}^2}(o,r+w)\right)\mathrm{d}\,z_1 \\ & = & p^2\lambda^2 \left(\frac{2\vartheta}{2\pi}\right) \cdot \text{area}_{\mathbb{H}^2}\left(B_{\mathbb{H}^2}(o,r+w)\right)^2 \\ & \leq & 1000\lambda^4\vartheta e^{2r+2w} \\ & = & 1000\vartheta e^{2w}, \end{array}$$

using rotational symmetry of the hyperbolic area measure in the third line; that $\text{area}_{\mathbb{H}^2}(B_{\mathbb{H}^2}(o,x)) = 2\pi(\cosh(x)-1) \leq \pi e^x$ and the bound on p in the fifth line; and the choice of r in the last. ■

Lemma 22 *For every $w, \lambda, h > 0$ and all $p \leq 10\lambda$ the following holds, setting $r := 2\log(1/\lambda)$. Writing*

$$X_{\mathbf{IV}} := \left| \left\{ (z_1, z_2) \in \mathcal{Z}_b \times \mathcal{Z}_b : \begin{array}{l} z_1 \in B_{\mathbb{H}^2}(o, r+w), \ \textit{and;} \\ z_1 \neq z_2, \ \textit{and;}, \\ \operatorname{dist}_{\mathbb{H}^2}(z_1, z_2) < h. \end{array} \right\} \right|,$$

we have

$$\mathbb{E}X_{\mathbf{IV}} \leq 1000\lambda^2 e^{w+h}.$$

Proof. Applying Corollary 5 again, we have

$$\mathbb{E}X_{\mathbf{IV}} = p^2\lambda^2 \int_{\mathbb{D}} \int_{\mathbb{D}} \mathbb{E}\left[g(z_1, z_2, \mathcal{Z} \cup \{z_1, z_2\})\right] f(z_1) f(z_2) \,\mathrm{d}\, z_2 \,\mathrm{d}\, z_2,$$

where f is given by (2) and this time

$$g(u_1, u_2, \mathcal{U}) = 1_{\left\{ \begin{array}{l} u_1 \in B_{\mathbb{H}^2}(o, r+w), \ \text{and;} \\ u_1 \neq u_2, \ \text{and;}, \\ \operatorname{dist}_{\mathbb{H}^2}(u_1, u_2) < h. \end{array} \right\}}.$$

In other words

$$\begin{aligned} \mathbb{E}X_{\mathbf{IV}} &= p^2\lambda^2 \int_{B_{\mathbb{H}^2}(o, r+w)} \int_{B_{\mathbb{H}^2}(z_1, h)} f(z_1) f(z_2) \,\mathrm{d}\, z_2 \,\mathrm{d}\, z_1 \\ &= p^2\lambda^2 \cdot \operatorname{area}_{\mathbb{H}^2}\left(B_{\mathbb{H}^2}(o, r+w)\right) \cdot \operatorname{area}_{\mathbb{H}^2}\left(B_{\mathbb{H}^2}(o, h)\right) \\ &\leq 1000\lambda^2 e^{w+h}, \end{aligned}$$

again using that $\operatorname{area}_{\mathbb{H}^2}(B_{\mathbb{H}^2}(o, x)) = 2\pi(\cosh x - 1) \leq \pi e^x$ and the choice of r. ■

Lemma 23 *There exists a $\lambda_0 > 0$ such that, for all $0 < \lambda < \lambda_0$, all $w \geq 0$ and all $p \leq 10\lambda$, the following holds, setting $r := 2\log(1/\lambda)$. Writing*

$$X_{\mathbf{V}} := \left| \left\{ z \in \mathcal{Z}_b : \begin{array}{l} \operatorname{dist}_{\mathbb{H}^2}(z, o) \geq r + w, \ \textit{and;} \\ \exists \textit{ a disk } B \textit{ with } o, z \in \partial B \textit{ and } \mathcal{Z} \cap B = \emptyset. \end{array} \right\} \right|,$$

we have

$$\mathbb{E}X_{\mathbf{V}} \leq 1000 e^{w} e^{-e^{w/2}}.$$

In the proof we'll make use of the following definition and observations, that we'll reuse later. For $z_1, z_2 \in \mathbb{D}$ the *Gabriel disk* is the disk $B_{\mathrm{Gab}}(z_1, z_2) := B_{\mathbb{H}^2}(c, \operatorname{dist}_{\mathbb{H}^2}(z_1, z_2)/2)$ whose center c is the midpoint of the hyperbolic line segment between z_1 and z_2 and whose radius is half the hyperbolic distance between z_1 and z_2. Put differently, among all disks that have both z_1 and z_2 on their boundary, $B_{\mathrm{Gab}}(z_1, z_2)$ is the one of smallest hyperbolic radius.

The name "Gabriel disk" is in reference to the *Gabriel graph*, an object that has received some attention in the discrete and computational geometry literature. The Gabriel graph associated with a point set $V \subseteq \mathbb{R}^2$ is the subgraph of the Delaunay graph whose edges are all pairs $v_1, v_2 \in V$ for which the Euclidean disk whose center is the midpoint between v_1 and v_2 and whose radius is half their distance contains no other points of V.

The hyperbolic line segment between z_1 and z_2 splits $B_{\text{Gab}}(z_1, z_2)$ into two parts of equal hyperbolic area. We shall denote by $B^-_{\text{Gab}}(z_1, z_2)$ the part that is on the left as we travel from z_1 to z_2 along the hyperbolic line segment, and by $B^+_{\text{Gab}}(z_1, z_2)$ the one that is on the right. See Figure 10.

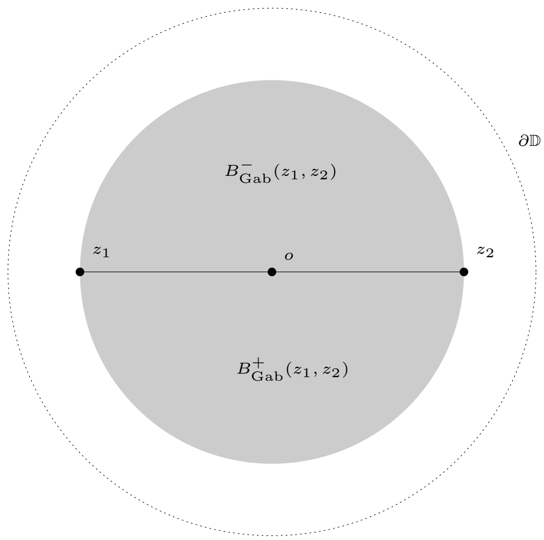

Figure 10: The Gabriel disk $B_{\text{Gab}}(z_1, z_2)$, in the special case when z_1, z_2 lie on the x-axis and their midpoint is the origin.

We'll repeatedly make use of the following straightforward observation.

(♠) For all $z_1, z_2 \in \mathbb{D}$ and every disk B such that $z_1, z_2 \in \partial B$, we have either $B^-_{\text{Gab}}(z_1, z_2) \subseteq B$ or $B^+_{\text{Gab}}(z_1, z_2) \subseteq B$ (or both).

(This is easily seen by applying a suitable isometry that maps z_1, z_2 to the x-axis and their midpoint to the origin, so that $B^-_{\text{Gab}}, B^+_{\text{Gab}}$ are mapped to ordinary, Euclidean half-disks – as in Figure 10.)

In light of observation **(♠)**, in order to prove Lemma 23 it suffices to prove the following statement, which we separate out as a lemma for convenient future reference.

Lemma 24 *There exists a* $\lambda_0 > 0$ *such that, for all* $0 < \lambda < \lambda_0$*, all* $w \geq 0$ *and all* $p \leq 10\lambda$*, the following holds, setting* $r := 2\log(1/\lambda)$*. Writing*

$$\tilde{X}_{\mathbf{V}} := \left| \left\{ z \in \mathcal{Z}_b : \begin{array}{l} \text{dist}_{\mathbb{H}^2}(z, o) \geq r + w, \text{ and;} \\ B^-_{Gab}(z, o) \cap \mathcal{Z} = \emptyset \text{ or } B^+_{Gab}(z, o) \cap \mathcal{Z} = \emptyset. \end{array} \right\} \right|,$$

we have

$$\mathbb{E}\tilde{X}_{\mathbf{V}} \leq 1000 e^w e^{-e^{w/2}}.$$

Proof of Lemma 24. We let $\lambda_0 > 0$ be a small constant, to be determined in the course of the proof. By Corollary 5

$$\mathbb{E}\tilde{X}_{\mathbf{V}} = p\lambda \int_{\mathbb{D}} \mathbb{E}\left[g(z, \mathcal{Z} \cup \{z\})\right] f(z) \,\mathrm{d}\, z,$$

where f is again given by (2) and

$$g(u, \mathcal{U}) := 1_{\left\{ \begin{array}{l} \mathrm{dist}_{\mathbb{H}^2}(u,o) \geq r+w, \text{ and;} \\ B^-_{\mathrm{Gab}}(u,o) \cap \mathcal{U} = \emptyset \text{ or } B^+_{\mathrm{Gab}}(u,o) \cap \mathcal{U} = \emptyset. \end{array} \right\}}.$$

We have

$$\begin{aligned}
\mathbb{E}\tilde{X}_{\mathbf{V}} &= p\lambda \int_{\mathbb{D}\setminus B_{\mathbb{H}^2}(o,r+w)} \mathbb{P}\left(B^-_{\mathrm{Gab}}(o,z) \cap \mathcal{Z} = \emptyset \text{ or } B^+_{\mathrm{Gab}}(o,z) \cap \mathcal{Z} = \emptyset\right) f(z) \,\mathrm{d}\, z \\
&\leq 10\lambda^2 \int_{\mathbb{D}\setminus B_{\mathbb{H}^2}(o,r+w)} 2\exp\left[-\frac{1}{2}\cdot\lambda\cdot \mathrm{area}_{\mathbb{H}^2}\left(B_{\mathrm{Gab}}(o,z)\right)\right] f(z)\,\mathrm{d}\, z \\
&\leq 20\lambda^2 \int_{\mathbb{D}\setminus B_{\mathbb{H}^2}(o,r+w)} \exp\left[-\lambda e^{\mathrm{dist}_{\mathbb{H}^2}(o,z)/2}\right] f(z)\,\mathrm{d}\, z,
\end{aligned}$$

where in the last line we use that λ_0 was chosen sufficiently small; that $2\pi(\cosh x - 1) = (1+o_x(1))\pi e^x$ as $x \to \infty$ so that $2\pi(\cosh x - 1) \geq 2e^x$ for x sufficiently large; that $\mathrm{dist}_{\mathbb{H}^2}(o,z) \geq r+w$; and that $r \geq 2\ln(1/\lambda_0)$.

Switching to hyperbolic polar coordinates (i.e. $z(\alpha,\rho) = (\cos(\alpha)\cdot\tanh(\rho/2), \sin(\alpha)\cdot\tanh(\rho/2))$) we find

$$\begin{aligned}
\mathbb{E}\tilde{X}_{\mathbf{V}} &\leq 20\lambda^2 \int_{r+w}^{\infty}\int_0^{2\pi} e^{-\lambda e^{\rho/2}} \sinh(\rho)\,\mathrm{d}\,\alpha\,\mathrm{d}\,\rho \\
&= 20\lambda^2 \int_{r+w}^{\infty} e^{-\lambda e^{\rho/2}} 2\pi\sinh(\rho)\,\mathrm{d}\,\rho \\
&\leq 20\pi\lambda^2 \int_{r+w}^{\infty} e^{-\lambda e^{\rho/2}} e^{\rho}\,\mathrm{d}\,\rho \\
&= 40\pi \int_{\lambda e^{(r+w)/2}}^{\infty} e^{-u} u\,\mathrm{d}\, u \\
&= 40\pi\left(\lambda e^{(r+w)/2} + 1\right)\exp\left[-\lambda e^{(r+w)/2}\right] \\
&= 40\pi\left(e^{w/2}+1\right) e^{-e^{w/2}} \\
&\leq 1000 e^{w} e^{-e^{w/2}},
\end{aligned}$$

using the substitution $u = \lambda e^{\rho/2}$ (so that $\mathrm{d}\,\rho = \frac{2\,\mathrm{d}\,u}{u}$) in the third line ■

Lemma 25 *There exists a $\lambda_0 > 0$ such that for all $0 < \lambda < \lambda_0$, all $w \geq 0$ and all $p \leq 10\lambda$ the following holds, setting $r := 2\log(1/\lambda)$. Writing*

$$X_{\mathbf{VI}} := \left|\left\{ z \in \mathcal{Z}_b : \begin{array}{l} r-w < \mathrm{dist}_{\mathbb{H}^2}(z,o) < r+w, \textit{ and;} \\ \exists \textit{ a disk } B \textit{ with } o,z \in \partial B, \mathcal{Z}\cap B = \emptyset \textit{ and } \mathrm{diam}_{\mathbb{H}^2}(B) \geq r+w. \end{array}\right\}\right|,$$

we have

$$\mathbb{E}X_{\mathbf{VI}} \leq 1000 e^{w} e^{-e^{w/2}}.$$

In the proof we'll make use of the following definition and observations, that we'll reuse later. For $z_1, z_2 \in \mathbb{D}$ and $\rho > 0$ with $\text{dist}_{\mathbb{H}^2}(z_1, z_2) < \rho$, there exist precisely two disks B such that $z_1, z_2 \in \partial B$ and $\text{diam}_{\mathbb{H}^2}(B) = \rho$. We let $DD(z_1, z_2, \rho)$ denote the union of these two disks. (The notation DD stands for "double disk".) The hyperbolic line segment between z_1 and z_2 splits $DD(z_1, z_2, \rho)$ into two parts of equal hyperbolic area. We denote by $DD^-(z_1, z_2, \rho)$ the part that is on the left as we travel from z_1 to z_2 along the hyperbolic line segment between them, and by $DD^+(z_1, z_2, \rho)$ the part on the right. See Figure 11.

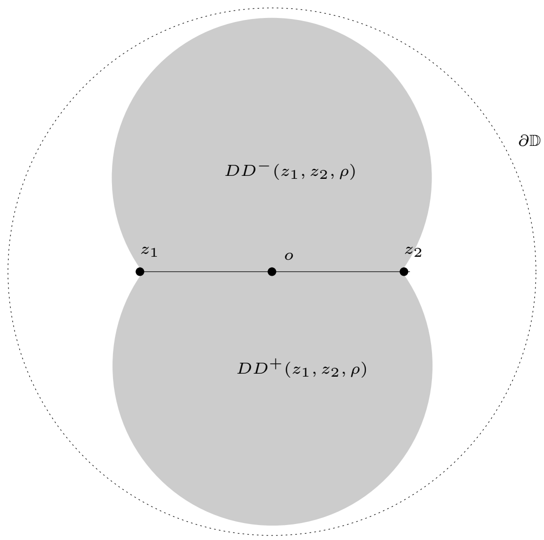

Figure 11: The set $DD(z_1, z_2, \rho)$, in the special case when z_1, z_2 lie on the x-axis with the origin o as their midpoint.

Another observation that we'll use repeatedly is:

(♣) For all $z_1 \neq z_2 \in \mathbb{D}$ and $\rho > \text{dist}_{\mathbb{H}^2}(z_1, z_2)$ and every hyperbolic disk B such that $z_1, z_2 \in \partial B$ and $\text{diam}_{\mathbb{H}^2}(B) \geq \rho$, we have either $DD^-(z_1, z_2, \rho) \subseteq B$ or $DD^+(z_1, z_2, \rho) \subseteq B$ (or both).

(Again this is easily seen by applying a suitable isometry that maps z_1, z_2 to the x-axis and their midpoint to the origin, as in Figure 11. Fact 2 ensures that the image of the hyperbolic disk B is a Euclidean disk with the images of z_1, z_2 on its boundary.)

In light of observation **(♣)**, in order to prove Lemma 25 it suffices to prove the following statement, which we separate out as a lemma for convenient future reference.

Lemma 26 *There exists a* $\lambda_0 > 0$ *such that for all* $0 < \lambda < \lambda_0$*, all* $w \geq 0$ *and all* $p \leq 10\lambda$ *the following holds, setting* $r := 2\log(1/\lambda)$*. Writing*

$$\tilde{X}_{\mathbf{VI}} := \left| \left\{ z \in \mathcal{Z}_b : \begin{array}{l} r - w < \text{dist}_{\mathbb{H}^2}(z, o) < r + w, \text{ and;} \\ DD^-(o, z, r+w) \cap \mathcal{Z} = \emptyset \text{ or } DD^+(o, z, r+w) \cap \mathcal{Z} = \emptyset. \end{array} \right\} \right|,$$

we have

$$\mathbb{E}\tilde{X}_{\mathbf{VI}} \leq 1000e^{w}e^{-e^{w/2}}.$$

Proof of Lemma 26. Applying Corollary 5, we have

$$\mathbb{E}\tilde{X}_{\mathbf{VI}} = p\lambda \int_{B_{\mathbb{H}^2}(o,r+w)} \mathbb{P}(DD^{-}(o,z,r+w) \cap \mathcal{Z} = \emptyset \text{ or } DD^{-}(o,z,r+w) \cap \mathcal{Z} = \emptyset) f(z)\, \mathrm{d}\, z,$$

where f is as given by (2). Next we point out that, by symmetry

$$\mathrm{area}_{\mathbb{H}^2}(DD^{-}(o,z,r{+}w)) = \mathrm{area}_{\mathbb{H}^2}(DD^{+}(o,z,r{+}w)) \geq \frac{1}{2}\,\mathrm{area}_{\mathbb{H}^2}(B_{\mathbb{H}^2}(o,(r{+}w)/2)) \geq e^{(r+w)/2},$$

where the last inequality holds provided we chose λ_0 sufficiently large, since $r \geq 2\ln(1/\lambda_0)$ and $\mathrm{area}_{\mathbb{H}^2}(B_{\mathbb{H}^2}(0,x)) = 2\pi(\cosh x - 1) = (1 + o_x(1))\pi e^x$ as $x \to \infty$.

Hence

$$\begin{aligned} \mathbb{E}\tilde{X}_{\mathbf{VI}} &\leq p\lambda\, \mathrm{area}_{\mathbb{H}^2}(B_{\mathbb{H}^2}(0,r+w)) \cdot 2e^{-\lambda e^{(r+w)/2}} \\ &\leq 1000\lambda^2 e^{r+w} e^{-\lambda e^{(r+w)/2}} \\ &= 1000e^{w}e^{-e^{w/2}}, \end{aligned}$$

using the choice of r. ■

We are now ready to prove the following statement.

Proposition 27 *For every $0 < \varepsilon < \frac{1}{1000}$ there exist $w, \vartheta > 0$ such that the following holds. For every $h > 0$ there exists a $\lambda_0 = \lambda_0(\varepsilon, w, \vartheta, h)$ such that, for all $0 < \lambda < \lambda_0$, setting $p := (1+\varepsilon)(\pi/3)\lambda$ and $r := 2\ln(1/\lambda)$, we have that*

$$\mathbb{E}X > 1.$$

(With X as defined in (11)*.)*

Proof. As pointed out right after the definition of X, its probability distribution is the same for any choice of $u \neq v \in \mathbb{D}$. For definiteness we take $u = o$ and $v = (1/2, 0)$. We let $h > 0$ be an arbitrary constant. We let $w, \lambda_0 > 0$ be large constants, and $\vartheta > 0$ a small constant, to be determined in the course of the proof.

Let us denote by D_b the number of black cells adjacent to the typical cell. (Recall that "typical cell" refers to the cell of the origin in the Voronoi tessellation for $\mathcal{Z} \cup \{o\}$.) By Isokawa's formula

$$\mathbb{E}D_b = p\mathbb{E}D = p \cdot \left(6 + \frac{3}{\pi\lambda}\right) > 1 + \varepsilon,$$

using the choice of p in the inequality. Next, we point out that

$$X \geq D_b - \left(X_{\mathbf{I}} + X_{\mathbf{II}} + X_{\mathbf{III}} + X_{\mathbf{IV}} + X_{\mathbf{V}} + X_{\mathbf{VI}}\right).$$

with $X_{\mathbf{I}}$–$X_{\mathbf{VI}}$ as defined in Lemmas 19–25. Applying these lemmas we see that, provided we chose λ_0 sufficiently small,

$$\mathbb{E}X > 1 + \varepsilon - 1000 \cdot \left(e^{-w} + \vartheta e^{w} + \vartheta e^{2w} + \lambda_0^2 e^{w+h} + 2e^{w} e^{-e^{w/2}} \right) > 1 + \varepsilon/2,$$

where the second inequality holds provided we chose the constant w sufficiently large and the constants ϑ, λ_0 sufficiently small. (To be more explicit : we can for instance first choose w such that $e^{-w}, e^{w}e^{-e^{w/2}} < \varepsilon/10^5$ and then we choose ϑ such that $\vartheta e^{2w} < \varepsilon/10^5$ and finally choose λ_0 such that $\lambda_0^2 e^{w+h} < \varepsilon/10^5$ – and at the same time λ_0 is small enough so that Lemma 23 and 25 apply.) ■

As pointed out earlier, Proposition 17, the upper bound in our main theorem, follows directly from Propositions 18 and 27.

3.3 The lower bound

Here we will show the following proposition, which constitutes the remaining half of our main result.

Proposition 28 *For every $\varepsilon > 0$, there exists a $\lambda_0 = \lambda_0(\varepsilon) > 0$ such that for all $0 < \lambda < \lambda_0$, we have $p_c(\lambda) \geq (1 - \varepsilon) \cdot (\pi/3) \cdot \lambda$.*

In the proof of Proposition 17 we could in a sense afford to "ignore" some adjacencies in the Voronoi tessellation. The supercritical Galton-Watson tree we constructed in the proof of Proposition 18 uses only adjacencies of convenient lengths and is such that if two edges share an endpoint, the angle they make is not too small.

Now we need to show that no infinite black component exists almost surely, for λ sufficiently small and $p = (1 - \varepsilon) \cdot (\pi/3) \cdot \lambda$. We cannot a priori exclude the possibility that an infinite component exists but every infinite connected subgraph contains (has to contain) "unusual" edges. In particular, a hypothetical infinite component might exist that does not contain an infinite (r, w, ϑ)-tree.

To make our life easier, we adopt a more generous notion of adjacency, in the form of *pseudopaths.*

Definition 29 *Given parameters $r, w_1, w_2, \vartheta > 0$, we say a (finite or infinite) sequence $u_0, u_1, \dots \in \mathbb{D}$ of distinct points is a* pseudopath *(wrt. r, w_1, w_2, ϑ and $\mathcal{Z}$) if one of the following holds for each $i \geq 1$:*

- **I** $r - w_1 < \mathrm{dist}_{\mathbb{H}^2}(u_{i-1}, u_i) < r + w_2, \angle u_{i-2}u_{i-1}u_i \geq \vartheta$ *and one of the following holds. There exists a disk B such that $u_{i-1}, u_i \in \partial B, B \cap \mathcal{Z} = \emptyset$ and* $\mathrm{diam}_{\mathbb{H}^2}(B) < r + w_2$*, or; $DD^-(u_{i-1}, u_i, r + w_2) \cap \mathcal{Z} = \emptyset$, or; $DD^+(u_{i-1}, u_i, r + w_2) \cap \mathcal{Z} = \emptyset$.*
- **II** $\mathrm{dist}_{\mathbb{H}^2}(u_{i-1}, u_i) \leq r - w_1$*;*
- **III** $r - w_1 < \mathrm{dist}_{\mathbb{H}^2}(u_{i-1}, u_i) < r + w_2$ *and $\angle u_{i-2}u_{i-1}u_i < \vartheta$;*
- **IV** $\mathrm{dist}_{\mathbb{H}^2}(u_{i-1}, u_i) \geq r + w_2$ *and either $B^-_{Gab}(u_{i-1}, u_i) \cap \mathcal{Z} = \emptyset$, or $B^+_{Gab}(u_{i-1}, u_i) \cap \mathcal{Z} = \emptyset$, (or both).*

(Of course the demand that $\angle u_{i-2}u_{i-1}u_i \geq \vartheta$ in **I** only applies when $i \geq 2$, and similarly the case **III** can only occur when $i \geq 2$.) The length of a finite pseudopath $P = u_0, u_1, \dots, u_n$ is n, the number of points minus one, and we use the term *pseudo-edge* for a consecutive pair of points u_{i-1}, u_i on a pseudopath. In particular pseudopaths of length one are pseudo-edges, but not all pseudo-edges are pseudopaths of length one (because of **III** which can only apply when the pseudopath has length ≥ 2). As in the case of ordinary paths, we denote by $V(P)$ the set of vertices of the pseudopath P.

With each pair of points $u, v \in \mathbb{D}$ of a pseudopath we associate a "certificate" $\mathrm{cert}(u,v) \subseteq \mathbb{D}$. This will be a region such that in order to verify that uv is a pseudo-edge, in addition to the location of the previous point of the pseudopath if uv is part of a pseudopath of length ≥ 2, only $\mathcal{Z} \cap \mathrm{cert}(u,v)$ is relevant. Specifically, we set

$$\mathrm{cert}(u,v) := \begin{cases} DD(u,v,r+w_2) & \text{if } r - w_1 < \mathrm{dist}_{\mathbb{H}^2}(u,v) < r + w_2; \\ B_{\mathrm{Gab}}(u,v) & \text{if } \mathrm{dist}_{\mathbb{H}^2}(u,v) \geq r + w_2; \\ \emptyset & \text{otherwise.} \end{cases}$$

(Note that if $\mathrm{dist}_{\mathbb{H}^2}(u,v) < r + w_2$ and B is a disk with $u, v \in \partial B$ and $\mathrm{diam}_{\mathbb{H}^2}(B) < r + w_2$ then $B \subseteq DD(u,v,r+w_2)$. Also note that if $\mathrm{dist}_{\mathbb{H}^2}(u,v) \geq r - w_1$ then $\mathrm{cert}(u,v) \supseteq B_{\mathrm{Gab}}(u,v)$.) For notational convenience we also set

$$\mathrm{cert}(u_0, \dots, u_k) := \mathrm{cert}(u_0, u_1) \cup \dots \cup \mathrm{cert}(u_{k-1}, u_k),$$

for every sequence $u_0, \dots, u_k \in \mathbb{D}$.

Pseudo-edges of type **I** will be called *good*, and all other types of pseudo-edges are *bad*. A pseudopath is good if all its pseudo-edges are good.

A finite pseudopath $P = u_0, \dots, u_k$ is called a *chunk* if the final pseudo-edge $u_{k-1}u_k$ is bad and all other pseudo-edges are good. So a chunk can for instance consist of a single bad pseudo-edge.

Definition 30 *A* linked sequence of chunks *is a (finite or infinite) sequence* $P_1, P_2, \dots$ *of chunks such that*

- **(i)** $V(P_i) \cap V(P_j) = \mathrm{cert}(P_i) \cap \mathrm{cert}(P_j) = \emptyset$ *if* $|i - j| > 1$*, and;*
- **(ii)** *For each* $i \geq 2$*, writing* $P_{i-1} = u_0^{i-1}, \dots, u_{k_{i-1}}^{i-1}$ *and* $P_i = u_0^i, \dots, u_{k_i}^i$*, we have*

$$\begin{aligned} &\{u_1^i, \dots, u_{k_i}^i\} \cap V(P_{i-1}) = \emptyset, \\ &\mathrm{cert}(u_1^i, \dots, u_{k_i}^i) \cap \mathrm{cert}(P_{i-1}) = \emptyset, \\ &\mathrm{dist}_{\mathbb{H}^2}(u_j^i, \mathrm{cert}(P_{i-1})) \geq \tfrac{r}{1000} \text{ for all } 1 \leq j \leq k_i, \end{aligned}$$

  and in addition one of the following holds:

  - **a)** $u_0^i = u_{k_{i-1}}^{i-1}$*, or*
  - **b)** $u_0^i \notin V(P_{i-1})$ *and* $\mathrm{cert}(u_0^i, u_1^i) \cap \mathrm{cert}(P_{i-1}) \neq \emptyset$*, or*
  - **c)** $u_0^i \notin V(P_{i-1}), \mathrm{cert}(u_0^i, u_1^i) \cap \mathrm{cert}(P_{i-1}) = \emptyset$ *and* $\mathrm{dist}_{\mathbb{H}^2}(u_0^i, \mathrm{cert}(P_{i-1})) < r/1000$*.*

We'll say a linked sequence of chunks is infinite if the sum of the lengths of the chunks is infinite. In other words, an infinite linked sequence of chunks either consists of infinitely many finite chunks, or consists of finitely many chunks of which the last chunk has infinite length.

Proposition 31 *For every $\lambda > 0, 0 \leq p \leq 1$ and $r, w_1, w_2, \vartheta > 0$, almost surely, one of the following holds.*

(i) *The black component of o in the Voronoi tessellation for $\mathcal{Z} \cup \{o\}$ is finite, or;*

(ii) *There is a black, infinite, good pseudopath, or;*

(iii) *There is a black, infinite linked sequence of chunks starting from o.*

For clarity, we emphasize that the infinite pseudopath mentioned in item **(ii)** does not need to contain the origin o.

Our strategy for the proof of Proposition 28 is of course to show, after having established Proposition 31, that there is a choice of r, w_1, w_2, ϑ such that, for all small enough λ and all $p \leq (1-\varepsilon) \cdot (\pi/3) \cdot \lambda$, almost surely options **(ii)** and **(iii)** of Proposition 31 do not occur. This will imply that the black component of the origin in the Voronoi tessellation for $\mathcal{Z} \cup \{o\}$ is a.s. finite. A short argument will then show this also implies that, almost surely, all black components in the Voronoi tessellation for $\mathcal{Z}$ are finite.

For the proof of Proposition 31 we will use the following observation that is a straightforward consequence of the Slivniak-Mecke formula and Isokawa's formula. We provide a proof for completeness.

Lemma 32 *For every $\lambda > 0$ and $0 \leq p \leq 1$, almost surely, every Voronoi cell is adjacent to a finite number of other Voronoi cells.*

Proof. Let us write

$$I := |\{z \in \mathcal{Z} : \deg(z) = \infty\}| .$$

By the Slivniak-Mecke formula

$$\mathbb{E}I = \lambda \int_{\mathbb{D}} \mathbb{E}\left[g(z, \mathcal{Z} \cup \{z\})\right] f(z) \,\mathrm{d}\, u,$$

where f is as given by (2) and $g(u, \mathcal{U}) := 1_{\{\deg(u;\mathcal{U})=\infty\}}$ with $\deg(u;\mathcal{U})$ denoting the degree of u in the Delaunay graph of $\mathcal{U}$. By symmetry considerations, for every (fixed) $u \in \mathbb{D}$ we have

$$\mathbb{E}\left[\deg(u; \mathcal{Z} \cup \{u\})\right] = \mathbb{E}\left[\deg(o, \mathcal{Z} \cup \{o\})\right] = \mathbb{E}D = 6 + \frac{3}{\pi\lambda} < \infty,$$

where D denotes the typical degree and we apply Isokawa's formula. It follows that, for every $u \in \mathbb{D}$:

$$\mathbb{E}\left[g(u, \mathcal{Z} \cup \{u\})\right] = \mathbb{P}\left(\deg(u; \mathcal{Z} \cup \{u\}) = \infty\right) = 0.$$

Hence also $\mathbb{E}I = 0$, which implies that $I = 0$ almost surely. ■

We'll also need the following observation in the proof of Proposition 31.

Lemma 33 *For every $\lambda > 0$, almost surely, for every $x > 0$ the number of pseudo-edges $z_1 z_2$ with $z_1, z_2 \in \mathcal{Z}$ for which $\operatorname{cert}(z_1, z_2) \cap B_{\mathbb{H}^2}(o, x) \neq \emptyset$ is finite.*

Proof. Fix an arbitrary $x > 0$. Clearly the number of pseudo-edges z_1z_2 with $z_1, z_2 \in \mathcal{Z}$ for which $\text{dist}_{\mathbb{H}^2}(z_1, z_2) < r + w_2$ and $\text{cert}(z_1, z_2) \cap B_{\mathbb{H}^2}(o, x) \neq \emptyset$ is bounded by X^2 where $X := \mathcal{Z} \cap B_{\mathbb{H}^2}(o, x+2r+2w_2)$. Since $\mathbb{E}X = \lambda \cdot \text{area}_{\mathbb{H}^2}(B_{\mathbb{H}^2}(o, x+2r+2w_2)) < \infty$, the random variable X is finite almost surely.

Pseudo-edges z_1z_2 not counted by X^2 must satisfy $\text{dist}_{\mathbb{H}^2}(z_1, z_2) \geq r + w_2$ and hence $\text{cert}(z_1, z_2) = B_{\text{Gab}}(z_1, z_2)$ and either $B^-_{\text{Gab}}(z_1, z_1) \cap \mathcal{Z} = \emptyset$ or $B^+_{\text{Gab}}(z_1, z_2) \cap \mathcal{Z} = \emptyset$ (or both). By symmetry considerations, it suffices to count

$$Y := \left| \left\{ (z_1, z_2) \in \mathcal{Z}^2 : \begin{array}{l} \text{dist}_{\mathbb{H}^2}(z_1, o) > \text{dist}_{\mathbb{H}^2}(z_2, o), \text{ and;} \\ B_{\text{Gab}}(z_1, z_2) \cap B_{\mathbb{H}^2}(o, x) \neq \emptyset, \text{ and;} \\ B^-_{\text{Gab}}(z_1, z_1) \cap \mathcal{Z} = \emptyset \text{ or } B^+_{\text{Gab}}(z_1, z_2) \cap \mathcal{Z} = \emptyset. \end{array} \right\} \right|.$$

For each pair (z_1, z_2) counted by Y there is an $n \in \mathbb{N}$ such that $n - 1 \leq \text{dist}_{\mathbb{H}^2}(o, z_1) \leq n$. We must also have that $\text{dist}_{\mathbb{H}^2}(z_1, z_2) > \text{dist}_{\mathbb{H}^2}(o, z_1) - x$, since $\text{diam}(B_{\text{Gab}}(z_1, z_2)) = \text{dist}_{\mathbb{H}^2}(z_1, z_2)$ and $B_{\text{Gab}}(z_1, z_2) \cap B_{\mathbb{H}^2}(o, x) \neq \emptyset$.

Writing

$$Y_n := \left| \left\{ (z_1, z_2) \in \mathcal{Z}^2 : \begin{array}{l} \text{dist}_{\mathbb{H}^2}(z_1, o), \text{dist}_{\mathbb{H}^2}(z_2, o) \leq n \text{ and;} \\ \text{dist}_{\mathbb{H}^2}(z_1, z_2) > n - x - 1, \text{ and;} \\ B^-_{\text{Gab}}(z_1, z_1) \cap \mathcal{Z} = \emptyset \text{ or } B^+_{\text{Gab}}(z_1, z_2) \cap \mathcal{Z} = \emptyset. \end{array} \right\} \right|,$$

we thus have

$$\mathbb{E}Y \leq \sum_n \mathbb{E}Y_n.$$

Applying the Slivniak-Mecke formula, we find that

$$\mathbb{E}Y_n = \lambda^2 \int_{\mathbb{D}} \int_{\mathbb{D}} \mathbb{E}\left[g(z_1, z_2, \mathcal{Z} \cup \{z_1, z_2\})\right] f(z_1) f(z_2) \,\mathrm{d}\, z_1 \,\mathrm{d}\, z_2,$$

where f is as given by (2) and

$$g(u_1, u_2, \mathcal{U}) := 1_{\left\{ \begin{array}{l} \text{dist}_{\mathbb{H}^2}(u_1, o), \text{dist}_{\mathbb{H}^2}(u_2, o) \leq n, \\ \text{dist}_{\mathbb{H}^2}(u_1, u_2) > n - x - 1, \\ B^-_{\text{Gab}}(u_1, u_1) \cap \mathcal{U} = \emptyset \text{ or } B^+_{\text{Gab}}(u_1, u_2) \cap \mathcal{U} = \emptyset. \end{array} \right\}}.$$

For every $z_1, z_2 \in \mathbb{D}$ have

$$\begin{array}{rcl} \mathbb{E}\left[g(z_1, z_2, \mathcal{Z} \cup \{z_1, z_2\})\right] & \leq & 1_{\left\{ \begin{array}{l} \text{dist}_{\mathbb{H}^2}(z_1, o), \text{dist}_{\mathbb{H}^2}(z_2, o) \leq n, \\ \text{dist}_{\mathbb{H}^2}(z_1, z_2) > n - x - 1 \end{array} \right\}} \cdot 2e^{-\lambda \cdot \text{area}_{\mathbb{H}^2}(B_{\text{Gab}}(z_1, z_2))/2} \\ & \leq & 1_{\left\{\text{dist}_{\mathbb{H}^2}(z_1, o), \text{dist}_{\mathbb{H}^2}(z_2, o) \leq n\right\}} \cdot 2e^{-ce^{n/2}}, \end{array}$$

for a suitably chosen small constant $c = c(x, \lambda)$. (The last step in some more detail : for all z_1, z_2 for which $\text{dist}_{\mathbb{H}^2}(z_1, z_2) > n - x - 1$, we have $\text{area}_{\mathbb{H}^2}(B_{\text{Gab}}(z_1, z_2)) \geq 2\pi(\cosh((n - x - 1)/2) - 1)$. Since $\cosh(t) = (1/2 + o_t(1))e^t$ as $t \to \infty$, there is an n_0 such that $2\pi(\cosh((n - x - 1)/2) - 1)/2 \geq e^{(n-x-1)/2}$ for all $n \geq n_0$. We can now choose a small $0 < c < (\lambda/2) \cdot e^{-(x+1)/2}$ such that $2e^{-ce^{n/2}} \geq 1$ for all $n \leq n_0$.) It follows that

$$\begin{aligned}\mathbb{E}Y_n &\leq (\lambda \cdot \text{area}_{\mathbb{H}^2}(B_{\mathbb{H}^2}(o,n)))^2 \cdot 2e^{-ce^{n/2}} \\ &= \lambda^2 \cdot (2\pi)^2 \cdot (\cosh n - 1)^2 \cdot 2e^{-ce^{n/2}} \\ &\leq 2\pi^2\lambda^2 e^{2n} e^{-ce^{n/2}},\end{aligned}$$

and hence

$$\mathbb{E}Y \leq 2\pi^2\lambda^2 \sum_n e^{2n} e^{-ce^{n/2}} < \infty.$$

So Y is finite almost surely.

We have now shown that for every fixed $x > 0$ we have $\mathbb{P}(N_x < \infty) = 1$ where

$$N_x := |\{\text{pseudo-edges } z_1, z_2 \in \mathcal{Z} \text{ for which } \text{cert}(z_1, z_2) \cap B_{\mathbb{H}^2}(o,x) \neq \emptyset\}|.$$

Since $N_x \leq N_y$ whenever $x < y$ we also have

$$\mathbb{P}\left(\bigcap_{x>0} \{N_x < \infty\}\right) = \lim_{x\to\infty} \mathbb{P}(N_x < \infty) = 1,$$

which is what needed to be shown. ■

Proof of Proposition 31. We fix arbitrary $\lambda > 0, 0 \leq p \leq 1$ and $r, w_1, w_2, \vartheta > 0$. Since the typical degree has finite expectation by Isokawa's formula, almost surely, the cell of the origin in the Voronoi tessellation for $\mathcal{Z} \cup \{o\}$ is adjacent to at most finitely many other cells. For every point of $\mathcal{Z}$, the number of cells it is adjacent to in the Voronoi tessellation for $\mathcal{Z} \cup \{o\}$ is at most one more than the number of cells it is adjacent to in the tessellation for $\mathcal{Z}$. In particular, by Lemma 32, also in the Voronoi tessellation for $\mathcal{Z} \cup \{o\}$, almost surely, each cell is adjacent to at most finitely many other cells.

We consider an arbitrary realization of the Poisson process $\mathcal{Z}$ (in other words, a locally finite point set $\mathcal{Z} \subseteq \mathbb{D}$, partitioned into two parts $\mathcal{Z} = \mathcal{Z}_\text{b} \uplus \mathcal{Z}_\text{w}$) for which each Voronoi cell of $\mathcal{Z} \cup \{o\}$ is adjacent to finitely many others, and such that, for every $x > 0$, the ball $B_{\mathbb{H}^2}(o,x)$ intersects at most finitely many certificates of pseudo-edges. We will show that in such a realization either the black cluster of the origin is finite, or there exists an infinite, good, black pseudopath, or there exists an infinite linked sequence of chunks all of whose points are black (or more than one of the three options occurs).

Suppose thus that the origin o lies in an infinite black component (otherwise we are done). Since each cell is adjacent to finitely many others, there must exist an infinite (ordinary) path $P = z_0, z_1, z_2, \dots$ with $z_0 = o$ and $z_1, z_2, \dots \in \mathcal{Z}_b$ (distinct). This is of course also a pseudo-path. If only finitely many pseudo-edges of P are bad then we are done, as P will contain a black, infinite, good pseudopath. Hence from now on we will assume that P has infinitely many bad edges. In particular, we can find a $t_1 \geq 1$ such that the pseudo-edges $z_0z_1, z_1z_2, \dots, z_{t_1-2}z_{t_1-1}$ are good and $z_{t_1-1}z_{t_1}$ is bad. We set $P_1 := z_0, z_1, \dots, z_{t_1}$.

Let us say that an index i *interacts* with a subpath $Q = z_s, z_{s+1}, \dots, z_t$ of P if one of the following holds: $s \leq i \leq t$ or $\text{dist}_{\mathbb{H}^2}(z_i, \text{cert}(Q)) < r/1000$ or $\text{cert}(z_i, z_{i+1}) \cap \text{cert}(Q) \neq \emptyset$.

We now let $s_2 \geq t_1$ be the largest index that interacts with P_1. That s_2 exists (is finite) follows from the fact that

$$\bigcup_{u\in\text{cert}(P_1)} B_{\mathbb{H}^2}\left(u, \frac{r}{1000}\right) \subseteq B_{\mathbb{H}^2}(o,x),$$

for some $x > 0$. Since P contains infinitely many bad pseudo-edges, there exists a $t_2 > s_2$ such that the pseudo-edges $z_{s_2}z_{s_2+1}, \dots, z_{t_2-2}z_{t_2-1}$ are good and $z_{t_2-1}z_{t_2}$ is bad. We set $P_2 := z_{s_2}, z_{s_2+1}, \dots, z_{t_2}$. (A minor subtlety needs to be added here. When we say $z_{t_2-1}z_{t_2}$ is bad, we mean that it is bad wrt. the path $z_{s_2}, z_{s_2+1}, \dots, z_{t_2}$. We do this in order to handle the case where $z_{s_2}z_{s_2+1}$ is a type **III** pseudo-edge of P. If we took this single pseudo-edge as the path P_2, then P_2 would not be a chunk but a good pseudopath.)

We now let $s_3 \geq t_2$ be the largest index that interacts with P_2. Again s_3 exists as $\bigcup_{u\in\mathrm{cert}(P_2)} B_{\mathbb{H}^2}(u, r/1000)$ is contained in $B_{\mathbb{H}^2}(o, x)$ for some $x > 0$. Since P contains infinitely many bad edges, there exists $t_3 > s_3$ such that the pseudo-edges $z_{s_3}z_{s_3+1}, \dots, z_{t_3-2}z_{t_3-1}$ are good and $z_{t_3-1}z_{t_3}$ is bad (wrt. the path $z_{s_3}, z_{s_3+1}, \dots, z_{t_3}$). We set $P_3 := z_{s_3}, z_{s_3+1}, \dots, z_{t_3}$.

We continue defining $s_4, s_5, \dots$ and $t_4, t_5, \dots$ and $P_4, P_5, \dots$ analogously, producing an infinite linked sequence of chunks $P_1, P_2, \dots$.

(We point out that the choice of s_i guarantees that no index $j > s_i$ interacts with any of $P_1, \dots, P_{i-1}$ while s_i interacts with P_{i-1} but not $P_1, \dots, P_{i-2}$. From this it easily follows that the demands **(i)**, **(ii)** of Definition 30 are met.)

Since we considered an arbitrary realization of $\mathcal{Z}$ that satisfies two conditions that both hold almost surely, it follows that almost surely either the black cluster of the origin is finite or there exists an infinite linked sequence of chunks starting from the origin all of whose points are black. ■

The main thing that now remains, in order to prove Proposition 28, is to show that for a suitable choice of r, w_1, w_2, ϑ and for λ sufficiently small and $p = (1-\varepsilon)(\pi/3)\lambda$ with $0 < \varepsilon < 1$ a small constant, almost surely there is no black, infinite, good pseudopath and no black, infinite linked sequence of chunks starting from o.

As in the proof of the upper bound, from now on we will always take $r := 2\ln(1/\lambda)$. The parameters $w_1, w_2, \vartheta > 0$ will be constants that depend on ε and that we will choose appropriately in the end. We will use an inductive approach bounding the expected number of good pseudopaths, respectively linked sequences of chunks, of length n starting from the origin. With this in mind, we will prove a number of lemmas designed to deal with the different ways in which an additional point can be added to an existing good pseudopath, respectively linked sequence of chunks, of length $n-1$. But, first we need to derive some additional facts about hyperbolic geometry. Again, when reading the paper for the first time, the reader may wish to only read the definitions and lemma statements and skip over the proofs.

3.3.1 More geometric ingredients

Let us say that a sequence of points $u_0, u_1, \dots, u_k \in \mathbb{D}$ is a *pre-chunk* (wrt. r, w_1, w_2, ϑ) if they are placed in such a way that they will form a chunk whenever the random point set $\mathcal{Z}$ turns out favourably. That is, $u_0, u_1, \dots, u_k \in \mathbb{D}$ is a pre-chunk if $r - w_1 < \mathrm{dist}_{\mathbb{H}^2}(u_{i-1}, u_i) < r + w_2$ for $i = 1, \dots, k-1$ and $\angle u_{i-2}u_{i-1}u_i > \vartheta$ for all $2 \leq i \leq k-1$ and in addition $\mathrm{dist}_{\mathbb{H}^2}(u_{k-1}, u_k) \leq r - w_1$ or $\mathrm{dist}_{\mathbb{H}^2}(u_{k-1}, u_k) \geq r + w_2$ or $r - w_1 < \mathrm{dist}_{\mathbb{H}^2}(u_{k-1}, u_k) < r + w_2$ and $\angle u_{k-2}u_{k-1}u_k \leq \vartheta$. In particular, a pre-chunk is such that $u_0, \dots, u_{k-1}$ forms a (r, w, ϑ)-path with $w := \max(w_1, w_2)$.

If $r - w_1 < \mathrm{dist}_{\mathbb{H}^2}(u_{i-1}, u_i) < r + w_2$ and $\angle u_{i-2}u_{i-1}u_i > \vartheta$ (in case $i \geq 2$) for $i = 1, \dots, k$ then we speak of a *good pre-pseudopath*.

The first geometric observation in this section will be helpful for the situation when we want to add a new pseudo-edge to an existing pseudopath, and will also be used by later proofs in this section. It tells us that if we want to add a new pseudo-edge to a good pre-pseudopath, then either the intersection of the certificate of the new pseudo-edge and the certificates of the existing path is contained in a ball of constant radius (and hence has rather small area), or the new edge makes a small angle with the last edge of the existing pseudo-path. This will translate into useful bounds once we start estimating the expected number of linked sequences of chunks later on.

Lemma 34 *For every $w_1, w_2, \vartheta_1, \vartheta_2 > 0$ there are $r_0 = r_0(w_1, w_2, \vartheta_1, \vartheta_2)$ and $h = h(w_1, w_2, \vartheta_1, \vartheta_2)$ such that the following holds for all $r \geq r_0$. If $u_0, \dots, u_k$ is such that $u_0, \dots, u_{k-1}$ is a good pre-pseudopath wrt. r, w_1, w_2, ϑ_1 then either*

$$\mathrm{cert}(u_0, \dots, u_{k-1}) \cap \mathrm{cert}(u_{k-1}, u_k) \subseteq B_{\mathbb{H}^2}(u_{k-1}, h),$$

or

$$\angle u_{k-2}u_{k-1}u_k < \vartheta_2,$$

(or both).

Before giving the proof, we point out that by setting $\vartheta_1 = \vartheta_2$ we obtain:

Corollary 35 *For every $w_1, w_2, \vartheta > 0$ there are $r_0 = r_0(w_1, w_2, \vartheta)$ and $h = h(w_1, w_2, \vartheta)$ such that the following holds for all $r \geq r_0$. If $u_0, \dots, u_k$ is a good pre-pseudopath (wrt. r, w_1, w_2, ϑ) then*

$$\mathrm{cert}(u_0, \dots, u_{k-1}) \cap \mathrm{cert}(u_{k-1}, u_k) \subseteq B_{\mathbb{H}^2}(u_{k-1}, h).$$

Proof of Lemma 34. We fix $0 < \vartheta' < \min(\vartheta_2/2, 1/1000)$. For any $r > 0$, if $P = u_0, \dots, u_k$ is a pre-chunk wrt. r, w_1, w_2, ϑ_1 then $P' = u_0, \dots, u_{k-1}$ is a (r, w, ϑ_1)-path, setting $w := \max(w_1, w_2)$. We also point out that, since P' is a good pre-pseudopath, $\mathrm{cert}(u_{i-1}, u_i) = DD(u_{i-1}, u_i, r+w_2) \subseteq B_{\mathbb{H}^2}(u_{i-1}, r+w) \cap B_{\mathbb{H}^2}(u_i, r+w)$ for $i = 1, \dots, k-1$. By Proposition 10 there exist constants $r_0 = r_0(w, \vartheta_1, \vartheta')$ and $h = h(w, \vartheta_1, \vartheta')$ such that, whenever $r \geq r_0$, we have

$$\begin{aligned} \mathrm{cert}(u_0, \dots, u_{k-1}) &\subseteq \bigcup_{i=0}^{k-2} B_{\mathbb{H}^2}(u_i, r+w) = \bigcup_{v \in P_{u_{k-2} \setminus u_{k-1}}} B_{\mathbb{H}^2}(v, r+w) \\ &\subseteq B_{\mathbb{H}^2}(u_{k-1}, h) \cup \mathrm{sect}\left(u_{k-1}, u_{k-2}, \vartheta'\right). \end{aligned}$$

Let us write $s := \max\left(\mathrm{dist}_{\mathbb{H}^2}(u_{k-1}, u_k), r+w\right)$. By Lemma 16 (applied with $r = s$), we can assume without loss of generality that the constant h is such that:

$$\mathrm{cert}(u_{k-1}, u_k) \subseteq B_{\mathbb{H}^2}(u_k, s) \subseteq B_{\mathbb{H}^2}(u_{k-1}, h) \cup \mathrm{sect}\left(u_{k-1}, u_k, \vartheta'\right).$$

Since $\vartheta_2 > 2\vartheta'$, we have that if $\angle u_{k-2}u_{k-1}u_k \geq \vartheta_2$ then $\mathrm{sect}\left(u_{k-1}, u_{k-2}, \vartheta'\right) \cap \mathrm{sect}\left(u_{k-1}, u_k, \vartheta'\right) = \emptyset$. It follows that either $\angle u_{k-2}u_{k-1}u_k < \vartheta_2$ or $\mathrm{cert}(u_0, \dots, u_{k-1}) \cap \mathrm{cert}(u_{k-1}, u_k) \subseteq B_{\mathbb{H}^2}(u_{k-1}, h)$, as claimed. ■

Our next observation provides an upper bound on the number of edges in a pseudopath whose certificates intersect a given ball. This will be of use to us later on several times, for instance when we want to estimate or bound the expected number of pseudo-edges that can act as the start of a new chunk extending a given sequence of chunks.

Lemma 36 *For every $w_1, w_2, \vartheta > 0$ there exists $r_0 = r_0(w_1, w_2, \vartheta) > 0$ such that, for every $r \geq r_0$, every pre-chunk $u_0, u_1, \dots, u_k$ and every ball $B = B_{\mathbb{H}^2}(c, s)$, it holds that*

$$|\{1 \leq i \leq k : \text{cert}(u_{i-1}, u_i) \cap B \neq \emptyset\}| \leq \frac{4s}{r} + 10.$$

Proof. By Proposition 9 and the remark following the definition of pre-chunk, there exists an r_0, K such that if $r \geq r_0$ and $0 \leq i, j \leq k-1$ then

$$\text{dist}_{\mathbb{H}^2}(u_i, u_j) \geq |i-j| \cdot (r - K) \geq |i-j| \cdot (r/2),$$

for all sufficiently large r and all $1 \leq i, j \leq k$.

For $1 \leq \ell \leq k-1$, let us denote by c_ℓ^+, c_ℓ^- the centers of the two disks whose union is $\text{cert}(u_{\ell-1}, u_\ell) = DD(u_{\ell-1}, u_\ell, r + w_2)$. If $1 \leq i < j \leq k-1$ then

$$\begin{aligned} \text{dist}_{\mathbb{H}^2}(c_i^+, c_j^+), \text{dist}_{\mathbb{H}^2}(c_i^+, c_j^-), \text{dist}_{\mathbb{H}^2}(c_i^-, c_j^+), \text{dist}_{\mathbb{H}^2}(c_i^-, c_j^-) &\geq \text{dist}_{\mathbb{H}^2}(u_i, u_{j+1}) - 2\left(\tfrac{r+w_2}{2}\right) \\ &\geq (j+1-i) \cdot (r/2) - 2r \\ &= |i-j| \cdot (r/2) - (3/2) \cdot r, \end{aligned}$$

(the second inequality holding for r sufficiently large).

Now observe $B \cap \text{cert}(u_{\ell-1}, u_\ell) \neq \emptyset$ if and only if $\min\left(\text{dist}_{\mathbb{H}^2}(c, c_\ell^+), \text{dist}_{\mathbb{H}^2}(c, c_\ell^-)\right) < (r + w_2)/2 + s$. So if $B \cap \text{cert}(u_{i-1}, u_i) \neq \emptyset$ and $B \cap \text{cert}(u_{j-1}, u_j) \neq \emptyset$ then

$$\begin{aligned} 2r + 2s &\geq 2 \cdot \left(\tfrac{r+w_2}{2} + s\right) \\ &\geq \min\left(\text{dist}_{\mathbb{H}^2}(c_i^+, c_j^+), \text{dist}_{\mathbb{H}^2}(c_i^+, c_j^-), \text{dist}_{\mathbb{H}^2}(c_i^-, c_j^+), \text{dist}_{\mathbb{H}^2}(c_i^-, c_j^-)\right) \\ &\geq |i-j| \cdot (r/2) - (3/2) \cdot r, \end{aligned}$$

the first inequality holding provided r_0 was chosen sufficiently large. Rearranging, we find

$$|i - j| \leq \frac{4s}{r} + 7.$$

This shows that the number of $1 \leq i \leq k-1$ such that $\text{cert}(u_{i-1}, u_i) \cap B \neq \emptyset$ is at most $\frac{4s}{r} + 8$. Including also $\text{cert}(u_{k-1}, u_k)$, we obtain $\frac{4s}{r} + 9$ which is smaller than the bound in the statement of the lemma. ■

Using the previous lemma, we can now show that, for any ball whose radius is only a constant larger than $(r + w_2)/2$ (the radius of the two balls whose union is the certificate of a good pseudoedge), most of the area of the ball lies outside the certificate of any given pseudopath.

Lemma 37 *For every $\varepsilon, w_1, w_2, \vartheta > 0$ there exists $r_0 = r_0(\varepsilon, w_1, w_2, \vartheta) > 0$ such that, for every $r \geq r_0$, every good pre-pseudopath $u_0, u_1, \dots, u_k$ and every ball $B = B_{\mathbb{H}^2}(c, s)$ with radius $s \geq (r + w_2)/2 + 10 \ln 10 - \ln \varepsilon$, it holds that*

$$\text{area}_{\mathbb{H}^2}(B \setminus \text{cert}(u_0, \dots, u_k)) \geq (1 - \varepsilon) \cdot \text{area}_{\mathbb{H}^2}(B).$$

Proof. We let r_0 be a large constant, to be chosen more precisely during the proof.

By Lemma 36, we have

$$\begin{array}{rcl} \text{area}_{\mathbb{H}^2}(B \cap \text{cert}(u_0,\dots,u_k)) & \leq & \left(\frac{8s}{r}+20\right)\cdot \text{area}_{\mathbb{H}^2}(B_{\mathbb{H}^2}(o,(r+w_2)/2)) \\ & \leq & \left(\frac{8s}{r}+20\right)\cdot \pi e^{(r+w_2)/2}. \end{array}$$

As $\text{area}_{\mathbb{H}^2}(B) = 2\pi(\cosh s - 1) = (1+o_s(1))\cdot\pi e^s$, we have, provided we chose r_0 sufficiently large, that $\text{area}_{\mathbb{H}^2}(B) \geq \frac{1}{2}\pi e^s$. Hence

$$\frac{\text{area}_{\mathbb{H}^2}(B \cap \text{cert}(u_0,\dots,u_k))}{\text{area}_{\mathbb{H}^2}(B)} \leq \left((16/r)se^{-s} + 40e^{-s}\right)\cdot e^{(r+w_2)/2}.$$

Since the RHS is decreasing in s for $s \geq 1$, we have

$$\begin{array}{rcl} \frac{\text{area}_{\mathbb{H}^2}(B\cap\text{cert}(u_0,\dots,u_k))}{\text{area}_{\mathbb{H}^2}(B)} & \leq & \left(16\frac{(r+w_2)/2+10\ln 10-\ln\varepsilon}{r}+40\right)\cdot e^{-10\ln 10+\ln\varepsilon} \\ & \leq & \left(56+16\frac{w_2/2+10\ln 10-\ln\varepsilon}{r_0}\right)\cdot 10^{-10}\cdot\varepsilon \\ & < & \varepsilon, \end{array}$$

provided we chose r_0 sufficiently large. ■

Next we present another observation that will be helpful for the situation when we want to add a new pseudo-edge to an existing pseudopath. It tells us that one of three things must happen, each of which will translate into useful bounds once we start estimating the number of linked sequences of chunks later on.

Corollary 38 *For every $\varepsilon, w_1, w_2, \vartheta_1, \vartheta_2 > 0$ there exists $r_0 = r_0(\varepsilon, w_1, w_2, \vartheta_1, \vartheta_2) > 0$ such that, for every $r \geq r_0$ and every pre-chunk $u_0, u_1, \dots, u_k$ wrt. r, w_1, w_2, ϑ_1 we have that either*

$$\text{area}_{\mathbb{H}^2}(B_{Gab}(u_{k-1},u_k)\setminus\text{cert}(u_0,\dots,u_{k-1})) \geq (1-\varepsilon)\cdot\text{area}_{\mathbb{H}^2}(B_{Gab}(u_{k-1},u_k)),$$

or

$$\text{dist}_{\mathbb{H}^2}(u_{k-1},u_k) \leq r - w_1,$$

or

$$r - w_1 < \text{dist}_{\mathbb{H}^2}(u_{k-1},u_k) < r + w_2 + 20\ln 10 - 2\ln\varepsilon \text{ and } \angle u_{k-2}u_{k-1}u_k < \vartheta_2.$$

(or more than one of the above hold).

Proof. We let r_0 be a large constant, to be determined in the course of the proof and we let $r \geq r_0$ be arbitrary and we let $u_0,\dots,u_k$ be an arbitrary pre-chunk wrt. r, w_1, w_2, ϑ_1.

Of course there is nothing to prove if $\text{dist}_{\mathbb{H}^2}(u_{k-1},u_k) \leq r - w_1$. If $\text{dist}_{\mathbb{H}^2}(u_{k-1},u_k) \geq r + w_2 + 20\ln 10 - 2\ln\varepsilon$ then $B_{\text{Gab}}(u_{k-1},u_k)$ has radius $\geq (r+w_2)/2 + 10\ln 10 - \ln\varepsilon$ and we are done by Lemma 37 – assuming without loss of generality we chose r_0 larger that the bound provided in that lemma.

Let us thus assume $r - w_1 < \text{dist}_{\mathbb{H}^2}(u_{k-1},u_k) < r + w_2 + 20\ln 10 - \ln\varepsilon =: r + w_2'$. By Lemma 34, there are constants $r_0' = r_0'(w_1, w_2', \vartheta_1, \vartheta_2), h = h(w_1, w_2', \vartheta_1, \vartheta_2)$ such that if $r \geq r_0'$ then either

$$B_{\mathrm{Gab}}(u_{k-1},u_k)\cap\mathrm{cert}(u_0,\dots,u_{k-1})\subseteq\mathrm{cert}(u_{k-1},u_k)\cap\mathrm{cert}(u_0,\dots,u_{k-1})\subseteq B_{\mathbb{H}^2}(u_{k-1},h),$$

or

$$\angle u_{k-2}u_{k-1}u_k<\vartheta_2.$$

In the latter case we are clearly done. In the former case we note that

$$\frac{\mathrm{area}_{\mathbb{H}^2}(B_{\mathrm{Gab}}(u_{k-1},u_k)\cap\mathrm{cert}(u_0,\dots,u_{k-1}))}{\mathrm{area}_{\mathbb{H}^2}(B_{\mathrm{Gab}}(u_{k-1},u_k))}\le\frac{\cosh h-1}{\cosh\left(\frac{r_0-w_1}{2}\right)-1}<\varepsilon,$$

provided $r_0\ge r_0'$ was chosen sufficiently large. ■

For some of the remaining proofs needed for the lower bound, we'll use the following notion which may be of independent interest. For sets $A,B\subseteq\mathbb{D}$ we let

$$\mathrm{ahd}(A,B):=\sup_{x\in B}\mathrm{dist}_{\mathbb{H}^2}(x,A)=\sup_{x\in B}\inf_{y\in A}\mathrm{dist}_{\mathbb{H}^2}(x,y).$$

The abbreviation ahd stands for "asymmetric Hausdorff distance". As the reader may already have recognized, the Hausdorff distance between A,B equals $\max(\mathrm{ahd}(A,B),\mathrm{ahd}(B,A))$.

The next two lemmas are a preparation for the third one, Lemma 41. That lemma allows us to bound the area of intersection of a given ball with the certificates of a given (pre-)pseudopath. This will be of use to us when we estimate the expected number of linked sequences of chunks later on.

Lemma 39 *For every $\varepsilon>0$ there exists an $a=a(\varepsilon)$ such that the following holds. Let $B=B_{\mathbb{H}^2}(c_1,r_1)$ be a disk and $C=\partial B_{\mathbb{H}^2}(c_2,r_2)$ a hyperbolic circle such that* $\mathrm{ahd}(B,C)\ge a$. *Then we have $B\cap C\subseteq\mathrm{sect}\,(c_2,c_1,\varepsilon)$.*

Proof. The statement is obviously true if $B\cap C=\emptyset$. Similarly it is also true if B and C intersect in a single point (the common point would line on the line joining c_1 and c_2). So from now on we can assume this is not the case.

Let x_1,x_2 denote the two intersection points of the circles ∂B and C. It suffices to show that, provided $\mathrm{ahd}(B,C)\ge a$ and a is sufficiently large, $\alpha:=\angle c_1c_2x_1(=\angle c_1c_2x_2)\le\varepsilon$. For convenience we write $d:=\mathrm{dist}_{\mathbb{H}^2}(c_1,c_2)$. We next point out that

$$\mathrm{ahd}(B,C)=r_2-r_1+d.$$

(To see this we first note that by the triangle inequality $\mathrm{dist}_{\mathbb{H}^2}(z,c_1)\le\mathrm{dist}_{\mathbb{H}^2}(z,c_2)+\mathrm{dist}_{\mathbb{H}^2}(c_1,c_2)=r_2+d$ for all $z\in C$. In other words, $\mathrm{dist}_{\mathbb{H}^2}(z,B)=\max(0,\mathrm{dist}_{\mathbb{H}^2}(z,c_1)-r_1)\le\max(0,r_2-r_1+d)$ for all $z\in C$. Applying a suitable isometry if needed, we can assume without loss of generality that $c_2=o$ is the origin and c_1 lies on the negative x-axis. The point $p=(\tanh(r_2/2),0)\in C$ satisfies $\mathrm{dist}_{\mathbb{H}^2}(z,B)=\mathrm{dist}_{\mathbb{H}^2}(z,c_1)-r_1=r_2-r_1+d$, as c_1,c_2,p lies on the x-axis which is a hyperbolic line.)

By the hyperbolic law of cosines,

$$
\begin{aligned}
e^{r_1} &\geq \cosh(r_1) \\
&= \cosh(r_2)\cosh(d) - \sinh(r_2)\sinh(d)\cos(\alpha) \\
&\geq \frac{e^{r_2+d}}{4} \cdot (1-\cos(\alpha)).
\end{aligned}
$$

Hence

$$
\cos(\alpha) \geq 1 - 4e^{r_1-r_2-d} = 1 - 4e^{-\operatorname{ahd}(B,C)}.
$$

The statement follows by taking $a > -\ln\left(\frac{1-\cos\varepsilon}{4}\right)$. ■

Lemma 40 *We have*

$$
\lim_{a\to\infty} \inf \left\{ \frac{\operatorname{area}_{\mathbb{H}^2}(B_2 \backslash B_1)}{\operatorname{area}_{\mathbb{H}^2}(B_2)} : B_1, B_2 \subseteq \mathbb{D} \text{ (hyperbolic) disks, } \operatorname{ahd}(B_1,B_2) \geq a \right\} = 1.
$$

Proof. It suffices to show that for every $\varepsilon > 0$ there exists a_0 such that $\operatorname{ahd}(B_1,B_2) \geq a_0$ implies that $\operatorname{area}_{\mathbb{H}^2}(B_2 \setminus B_1) \geq (1-\varepsilon)\cdot \operatorname{area}_{\mathbb{H}^2}(B_2)$. We thus let a_0 be a large constant, to be determined in the course of the proof, and we let $B_1 = B_{\mathbb{H}^2}(c_1,r_1), B_2 = B_{\mathbb{H}^2}(c_2,r_2)$ be arbitrary disks with $a := \operatorname{ahd}(B_1,B_2) \geq a_0$.

If $r_2 \leq a/2$ then B_1 and B_2 are disjoint and we are done. For the remainder of the proof, we therefore assume $r_2 > a/2$.

Each of the circles $C_s := \partial B_{\mathbb{H}^2}(c_2,s)$ with $r_2 - \frac{a}{1000} \leq s \leq r_2$ satisfies $\operatorname{ahd}(B_1,C_s) \geq \left(\frac{999}{1000}\right)a$. Applying Lemma 39, having chosen a_0 sufficiently large, we can assume that $C_s \cap B_1 \subseteq \operatorname{sect}\left(c_2,c_1,\frac{\varepsilon}{1000}\right)$ for all $r_2 - \frac{a}{1000} \leq s \leq r_2$. It follows that

$$
B_1 \cap B_2 \subseteq B_{\mathbb{H}^2}\left(c_2, r_2 - \frac{a}{1000}\right) \cup \operatorname{sect}\left(c_2,c_1,\frac{\varepsilon}{1000}\right).
$$

Next we note that, by rotational symmetry of the hyperbolic area measure

$$
\frac{\operatorname{area}_{\mathbb{H}^2}\left(\operatorname{sect}\left(c_2,c_1,\frac{\varepsilon}{1000}\right)\cap B_2\right)}{\operatorname{area}_{\mathbb{H}^2}(B_2)} = \frac{\varepsilon}{1000\cdot\pi}. \tag{14}
$$

We also have

$$
\begin{aligned}
\frac{\operatorname{area}_{\mathbb{H}^2}(B_{\mathbb{H}^2}(c_2,r_2-\frac{a}{1000}))}{\operatorname{area}_{\mathbb{H}^2}(c_2,r_2)} &= \frac{\cosh(r_2-\frac{a}{1000})-1}{\cosh(r_2)-1} \\
&\leq \frac{\cosh(r_2-\frac{a}{1000})}{\cosh(r_2)} \\
&\leq \frac{\cosh(\frac{499}{1000}a)}{\cosh(a/2)} \\
&\leq \frac{\varepsilon}{1000},
\end{aligned} \tag{15}
$$

where the second inequality follows from the fact that $x \mapsto \cosh(x-y)/\cosh(x)$ is decreasing in x for $x \geq y$ and the last inequality holds provided we chose a_0 sufficiently large, and uses that $\cosh(\frac{499}{1000}a)/\cosh(a/2) \to 0$ as $a \to \infty$.

Combining (14) and (15), we see that

$$\text{area}_{\mathbb{H}^2}(B_1 \cap B_2) < \varepsilon \cdot \text{area}_{\mathbb{H}^2}(B_2),$$

provided we chose a_0 sufficiently large. This clearly implies the statement of the lemma. ■

Lemma 41 *For every $\varepsilon, w_1, w_2, \vartheta > 0$ there exist $r_0 = r_0(\varepsilon, w_1, w_2, \vartheta), a = a(\varepsilon, w, \vartheta) > 0$ such that, for every pre-chunk $u_0, \dots, u_k$ and every disk B with $\text{ahd}(\text{cert}(u_0, \dots, u_k), B) \geq a$ we have*

$$\text{area}_{\mathbb{H}^2}(B \setminus \text{cert}(u_0, \dots, u_k)) \geq (1 - \varepsilon)\, \text{area}_{\mathbb{H}^2}(B).$$

Proof. We fix a large constant $K > 0$, to be determined more precisely during the course of the proof. For convenience we'll write $C := \text{cert}(u_0, \dots, u_k)$ and $B = B_{\mathbb{H}^2}(c, s)$. We let $B_1 = B_{\mathbb{H}^2}(c_1, r_1), \dots, B_N = B_{\mathbb{H}^2}(c_N, r_N)$ be the balls that feature in the definition of $\text{cert}(u_0, u_1), \dots, \text{cert}(u_{k-1}, u_k)$. That is each B_i is either $B_{\text{Gab}}(u_{k-1}, u_k)$ or a disk of diameter $r + w_2$ with $u_{j-1}, u_j \in \partial B_i$ for some $1 \leq j \leq k$. In particular $C = B_1 \cup \dots \cup B_N$ and $N \leq 2k$.

By Lemma 40 we can take the constant a such that the assumption that $\text{ahd}(C, B) \geq a$ implies that $\text{area}_{\mathbb{H}^2}(B \setminus B_i) \geq (1 - \varepsilon/K)\, \text{area}_{\mathbb{H}^2}(B)$ for each $i = 1, \dots, N$.

We set $I := \{i : B_i \cap B \neq \emptyset\}$. If $|I| \leq K$ then the statement clearly holds. For the remainder of the proof we thus assume $|I| > K$.

By Lemma 36, assuming r_0 was chosen sufficiently large, we have $|I| \leq 8(s/r) + 20$ (each certificate is either a single ball or the union of two balls). Hence, provided we chose r_0 and K sufficiently large, $|I| > K$ implies $s \geq (K/10) \cdot r$. We have

$$\begin{aligned}
\text{area}_{\mathbb{H}^2}(B \setminus C) &\geq \text{area}_{\mathbb{H}^2}(B) - \textstyle\sum_{i \in I} \text{area}_{\mathbb{H}^2}(B \cap B_i) \\
&\geq \text{area}_{\mathbb{H}^2}(B) - \left(\tfrac{\varepsilon}{K}\right) \text{area}_{\mathbb{H}^2}(B) - \textstyle\sum_{i \in I : B_i \neq B_{\text{Gab}}(u_{k-1}, u_k)} \text{area}_{\mathbb{H}^2}(B_i) \\
&\geq (1 - \varepsilon/K) \cdot \text{area}_{\mathbb{H}^2}(B) - (8(s/r) + 20) \cdot \text{area}_{\mathbb{H}^2}(B_{\mathbb{H}^2}(o, (r + w_2)/2)) \\
&\geq (1 - \varepsilon/K) \cdot \text{area}_{\mathbb{H}^2}(B) - s \cdot \text{area}_{\mathbb{H}^2}(B_{\mathbb{H}^2}(o, (r + w_2)/2)),
\end{aligned}$$

the last inequality holding assuming we have chosen r_0 and K sufficiently large. Next we point out that

$$\begin{aligned}
\frac{s \cdot \text{area}\left(B_{\mathbb{H}^2}(o, (r + w_2)/2)\right)}{\text{area}_{\mathbb{H}^2}(B)} &= \frac{s \cdot (\cosh((r + w_2)/2) - 1)}{\cosh s - 1} \\
&\leq 1000 \cdot e^{(r + w_2)/2} \cdot s \cdot e^{-s} \\
&\leq 1000 \cdot e^{(r + w_2)/2} \cdot (K/10) \cdot r \cdot e^{-(K/10)r} \\
&= 1000 \cdot e^{w_2/2} \cdot (K/10) \cdot r \cdot e^{-\left(\frac{K-5}{10}\right)r} \\
&\leq 1000 \cdot e^{w_2/2} \cdot (K/10) \cdot r_0 \cdot e^{-\left(\frac{K-5}{10}\right)r_0} \\
&\leq \varepsilon/2,
\end{aligned}$$

provided r_0 and K were chosen sufficiently large, using in the second line that $\cosh x - 1 = (1 + o_x(1)) \cdot \frac{1}{2} \cdot e^x$ as $x \to \infty$; in the third line that $s \mapsto se^{-s}$ is decreasing for $s > 1$; and in the last line that $r \cdot e^{-cr} \to 0$ as $r \to \infty$ (for every $c > 0$). It follows that if $|I| > K$ then also

$$\text{area}_{\mathbb{H}^2}(B \setminus C) > (1 - \varepsilon) \cdot \text{area}_{\mathbb{H}^2}(B),$$

as claimed. ■

We conclude this subsection, with two lemmas giving conditions under which hyperbolic angles are small. Again, we will use these when estimating the expected number of linked sequences of chunks later on.

Lemma 42 *There is a universal constant $K > 0$ such that the following holds. If $B_1 = B_{\mathbb{H}^2}(c_1, r_1), B_2 = B_{\mathbb{H}^2}(c_2, r_2)$ and $p \in \partial B_2$ are such that $B_1 \cap B_2 \neq \emptyset$ then either $\text{dist}_{\mathbb{H}^2}(p, c_1) < r_1 + K$ or $\angle c_1 p c_2 < 10 \exp[(r_1 - \text{dist}_{\mathbb{H}^2}(p, c_1))/2]$.*

Proof. Let $K = K(\pi/2)$ be as provided by Lemma 11. For notational convenience we write $\alpha := \angle c_1 p c_2, \rho := \text{dist}_{\mathbb{H}^2}(c_1, p)$ and $d := \text{dist}_{\mathbb{H}^2}(c_1, c_2)$.

The disks B_1 and B_2 intersect if and only if $d < r_1 + r_2$. If $\alpha \geq \pi/2$ then Lemma 11 tells us that $d \geq \rho + r_2 - K$. It follows that if $\alpha \geq \pi/2$ and $B_1 \cap B_2 \neq \emptyset$ then $\rho < r_1 + K$.

Let us then assume $\alpha < \pi/2$. If $B_1 \cap B_2 \neq \emptyset$ then the hyperbolic cosine rule gives

$$\begin{array}{rcl} e^{r_1+r_2} & > & \cosh d \\ & = & \cosh(\rho)\cosh(r_2) - \cos(\alpha)\sinh(\rho)\sinh(r_2) \\ & \geq & \frac{1}{4}e^{\rho+r_2}(1-\cos\alpha) \\ & \geq & \frac{1}{4\pi}e^{\rho+r_2}\alpha^2, \end{array}$$

where we have used that $1 - \cos(\alpha) \geq \frac{\alpha^2}{\pi}$ for all $0 < \alpha < \pi/2$. Rearranging and taking square roots gives $\alpha < \sqrt{4\pi} \cdot e^{(r_1-\rho)/2} < 10e^{(r_1-\rho)/2}$. ∎

We'll use another rather straightforward consequence of the hyperbolic cosine rule:

Lemma 43 *For all $x_1, x_2 \in \mathbb{D}$ we have*

$$2\pi \cdot \exp\left[\frac{1}{2} \cdot (\text{dist}_{\mathbb{H}^2}(x_1, x_2) - \text{dist}_{\mathbb{H}^2}(o, x_1) - \text{dist}_{\mathbb{H}^2}(o, x_2))\right] > \angle x_1 o x_2.$$

Proof. For notational convenience, write $\rho_1 = \text{dist}_{\mathbb{H}^2}(o, x_1), \rho_2 = \text{dist}_{\mathbb{H}^2}(o, x_2)$ and $\gamma = \angle x_1 o x_2$. It suffices to show that $e^{\text{dist}_{\mathbb{H}^2}(x_1,x_2)} > \frac{1}{4\pi^2} \cdot e^{\rho_1+\rho_2} \cdot \gamma^2$.

If $\gamma < \pi/2$ then the hyperbolic cosine rule gives that

$$\begin{array}{rcl} e^{\text{dist}_{\mathbb{H}^2}(x_1,x_2)} & > & \cosh(\text{dist}_{\mathbb{H}^2}(x_1, x_2)) \\ & = & \cosh(\rho_1)\cosh(\rho_2) - \cos(\gamma)\sinh(\rho_1)\sinh(\rho_2) \\ & \geq & \frac{1}{4} \cdot e^{\rho_1+\rho_2} \cdot (1-\cos\gamma) \\ & \geq & \frac{1}{4\pi} \cdot e^{\rho_1+\rho_2} \cdot \gamma^2. \end{array}$$

(Using that $1 - \cos(\gamma) \geq \frac{\alpha^2}{\pi}$ for $0 < \gamma < \pi/2$ in the third line.)

If on the other hand $\gamma \in [\pi/2, \pi]$ then similarly

$$\begin{array}{rcl} e^{\text{dist}_{\mathbb{H}^2}(x_1,x_2)} & > & \cosh(\rho_1)\cosh(\rho_2) - \cos(\gamma)\sinh(\rho_1)\sinh(\rho_2) \\ & \geq & \cosh(\rho_1)\cosh(\rho_2) \\ & \geq & \frac{1}{4} \cdot e^{\rho_1+\rho_2} \\ & \geq & \frac{1}{4\pi^2} \cdot e^{\rho_1+\rho_2} \cdot \gamma^2. \end{array}$$

∎

3.3.2 Counting good pseudopaths

The next point of order is to bound the expected number of black, good pseudopaths starting from the origin. We plan to use an inductive approach and begin with a couple of lemmas designed for the situation where we add a good pseudo-edge to an existing, good pseudo-path.

Lemma 44 *For every $w_1, w_2, \vartheta > 0$ and $0 < \varepsilon < 1$ there exist $\lambda_0 = \lambda_0(\varepsilon, w_1, w_2, \vartheta)$ such that the following holds for all $0 < \lambda < \lambda_0$ and all $p \leq (1-\varepsilon) \cdot (\pi/3) \cdot \lambda$, setting $r := 2\log(1/\lambda)$.*

Let $u_0, \dots, u_k$ be an arbitrary good pre-pseudopath and, writing $C := \operatorname{cert}(u_0, \dots, u_k)$, define

$$Y_{\textbf{I-i}} := \left| \left\{ z \in \mathcal{Z}_b : \begin{array}{l} r - w_1 \leq \operatorname{dist}_{\mathbb{H}^2}(u_k, z) \leq r + w_2, \ \textit{and;} \\ \angle u_{k-1} u_k z > \vartheta, \ \textit{and;} \\ \exists \textit{ a disk } B \textit{ with } \mathcal{Z} \cap B \setminus C = \emptyset, \\ u_k, z \in \partial B \textit{ and } \operatorname{diam}_{\mathbb{H}^2}(B) < r + w_2. \end{array} \right\} \right|.$$

We have

$$\mathbb{E} Y_{\textbf{I-i}} \leq (1 - \varepsilon/2).$$

Before proceeding with the proof, let us clarify that in the above lemma we also allow $k = 0$. In that case the demand $\angle u_{k-1} u_k z > \vartheta$ of course does not apply.

Proof. We let λ_0 be a small constant to be determined in the course of the proof, and take an arbitrary $\lambda < \lambda_0$ and $p \leq (1-\varepsilon) \cdot (\pi/3) \cdot \lambda$. Once again we apply Corollary 5 to find that

$$\mathbb{E} Y_{\textbf{I-i}} = p\lambda \int_{\mathbb{D}} \mathbb{E}\left[g(z, \mathcal{Z} \cup \{z\}) \right] f(z) \, \mathrm{d}\, z,$$

where f is again as given by (2) and this time

$$g(u, \mathcal{U}) := 1_{\left\{ \begin{array}{l} r - w_1 \leq \operatorname{dist}_{\mathbb{H}^2}(u_k, u) \leq r + w_2, \ \text{and;} \\ \angle u_{k-1} u_k u > \vartheta, \ \text{and;} \\ \exists \text{ a disk } B \text{ with } \mathcal{U} \cap B \setminus C = \emptyset, \\ u_k, u \in \partial B \text{ and } \operatorname{diam}_{\mathbb{H}^2}(B) < r + w_2. \end{array} \right\}}.$$

By Corollary 35, assuming λ_0 was chosen sufficiently small, if $z \in \mathbb{D}$ is such that $\mathbb{E}\left[g(z, \mathcal{Z} \cup \{z\}) \right] \neq 0$ then $C \cap \operatorname{cert}(u_k, z) \subseteq B_{\mathbb{H}^2}(u_k, h)$.

Applying a suitable isometry if needed (which leaves the distribution of $\mathcal{Z}$ invariant), we can assume without loss of generality that $u_k = o$ is the origin. Recall that any disk B satisfying $o, z \in \partial B$ and $\operatorname{diam}_{\mathbb{H}^2}(B) < r + w$ satisfies $B \subseteq DD(o, z, r + w)$. So if $z \in \mathbb{D}$ is such that $r - w_1 \leq \operatorname{dist}_{\mathbb{H}^2}(0, z) \leq r + w_2$ and $\angle u_{k-1} o z > \vartheta$ then

$$\begin{aligned}
\mathbb{E}\left[g(z,\mathcal{Z}\cup\{z\})\right] &= \mathbb{P}\left(\begin{array}{l}\exists \text{ a disk } B \text{ with } \mathcal{Z}\cap B=\emptyset,\\ o,z\in\partial B \text{ and } \operatorname{diam}_{\mathbb{H}^2}(B)\le r+w_2\end{array}\Big|\mathcal{Z}\cap C=\emptyset\right)\\
&= \mathbb{P}\left(\begin{array}{l}\exists \text{ a disk } B \text{ with } \mathcal{Z}\cap B=\emptyset,\\ o,z\in\partial B \text{ and } \operatorname{diam}_{\mathbb{H}^2}(B)\le r+w_2\end{array}\Big|\mathcal{Z}\cap C\cap\operatorname{cert}(o,z)=\emptyset\right)\\
&\le \frac{\mathbb{P}\left(\exists \text{ a disk } B \text{ with } \mathcal{Z}\cap B=\emptyset \text{ and } o,z\in\partial B\right)}{\mathbb{P}\left(\mathcal{Z}\cap C\cap\operatorname{cert}(o,z)=\emptyset\right)}\\
&\le \frac{\mathbb{P}\left(\exists \text{ a disk } B \text{ with } \mathcal{Z}\cap B=\emptyset \text{ and } o,z\in\partial B\right)}{\mathbb{P}\left(\mathcal{Z}\cap B_{\mathbb{H}^2}(o,h)=\emptyset\right)}\\
&= \frac{\mathbb{E}\left[\hat{g}(z,\mathcal{Z}\cup\{z\})\right]}{\mathbb{P}(\mathcal{Z}\cap B_{\mathbb{H}^2}(o,h)=\emptyset)},
\end{aligned}$$

where $\hat{g}(u,\mathcal{U})=1$ if ou is an edge of the Delaunay graph for $\mathcal{U}\cup\{o\}$ and $\hat{g}(u,\mathcal{U})=0$ otherwise. It follows that

$$\begin{aligned}
\mathbb{E}Y_{\textbf{I-i}} &\le \frac{p\lambda}{\mathbb{P}(\mathcal{Z}\cap B_{\mathbb{H}^2}(o,h)=\emptyset)}\cdot\int_{\mathbb{D}}\mathbb{E}\left[\hat{g}(z,\mathcal{Z}\cup\{z\})\right]f(z)\,\mathrm{d}\,z\\
&= \frac{p}{\mathbb{P}(\mathcal{Z}\cap B_{\mathbb{H}^2}(o,h)=\emptyset)}\cdot\left(6+\frac{3}{\pi\lambda}\right)\\
&\le \frac{(1-\varepsilon)\cdot(1+2\pi\lambda)}{1-\lambda\cdot\operatorname{area}_{\mathbb{H}^2}(B_{\mathbb{H}^2}(o,h))},
\end{aligned}$$

where the second line follows by Isokawa's formula (and the Slivniak-Mecke formula to phrase the typical degree as an integral).

Since h is a constant we have $\lambda\cdot\operatorname{area}_{\mathbb{H}^2}(B_{\mathbb{H}^2}(o,h))\to 0$ as $\lambda\searrow 0$. So, provided we chose λ_0 sufficiently small, we have

$$\mathbb{E}Y_{\textbf{I-i}} < 1-\varepsilon/2,$$

as claimed. ■

Lemma 45 *For every $w_1,w_2,\vartheta>0$ there exists a $\lambda_0=\lambda_0(w_1,w_2,\vartheta)$ such that the following holds for all $\lambda<\lambda_0$ and all $p\le 10\lambda$, setting $r:=2\log(1/\lambda)$.*

Let $u_0,\dots,u_k$ be an arbitrary good pre-pseudopath and, writing $C:=\operatorname{cert}(u_0,\dots,u_k)$, define

$$Y_{\textbf{I-ii}} := \left|\left\{ z\in\mathcal{Z}_b : \begin{array}{l} r-w_1<\operatorname{dist}_{\mathbb{H}^2}(u_k,z)<r+w_2, \textit{ and;}\\ \angle u_{k-1}u_kz>\vartheta, \textit{ and;}\\ \textit{either } \mathcal{Z}\cap DD^-(u_k,z,r+w)\setminus C=\emptyset,\\ \textit{or } \mathcal{Z}\cap DD^+(u_k,z,r+w)\setminus C=\emptyset,\\ \textit{(or both).}\end{array}\right\}\right|.$$

We have

$$\mathbb{E}Y_{\textbf{I-ii}} \le 10^4\cdot e^{w_2}\cdot e^{-e^{w_2/2}}.$$

Proof. The proof is nearly identical to the proof of the previous lemma. Reasoning as the proof of Lemma 44, there is a constant $h=h(w_1,w_2,\vartheta)$ such that

$$\mathbb{E}Y_{\textbf{I-ii}} \leq \frac{\mathbb{E}\tilde{X}_{\textbf{VI}}}{1 - \lambda \cdot \text{area}_{\mathbb{H}^2}(B_{\mathbb{H}^2}(o,h))},$$

for all sufficiently small λ, where $\tilde{X}_{\textbf{VI}}$ is as in Lemma 26.

Since $1 - \lambda \cdot \text{area}_{\mathbb{H}^2}(B_{\mathbb{H}^2}(o,h)) > 1/10$ provided we chose λ_0 sufficiently small, the result follows from Lemma 26 ■

These last two lemmas allow us to prove the following statement using an inductive approach. Here and in the rest of the paper we write

$$G_k := |\{\text{black, good pseudopaths of length } k \text{ starting from } o\}|.$$

Lemma 46 *For every $0 < \varepsilon < 1$ and $w_1, w_2, \vartheta > 0$ there exists a $\lambda_0 = \lambda_0(\varepsilon, w_1, w_2, \vartheta)$ such that the following holds for all $0 < \lambda < \lambda_0$ and all $p \leq (1-\varepsilon)\cdot(\pi/3)\cdot\lambda$, setting $r := 2\ln(1/\lambda)$.*

$$\mathbb{E}G_k \leq \left(1 - \varepsilon/2 + 10^4 e^{w_2} e^{-e^{w_2/2}}\right)^k.$$

Proof. By Corollary 5 we have

$$\mathbb{E}G_k = (p\lambda)^n \int_{\mathbb{D}} \cdots \int_{\mathbb{D}} \mathbb{E}\left[g_k(z_1,\dots,z_k;\mathcal{Z}\cup\{z_1,\dots,z_k\})\right] f(z_1)\dots f(z_k)\,\mathrm{d}\,z_k\dots\mathrm{d}\,z_1, \tag{16}$$

where f is as given by (2) and g_k is the indicator function of the event that $o, z_1, \dots, z_k$ form a good pseudopath (wrt. the parameters r, w_1, w_2, ϑ and the point set $\mathcal{Z}\cup\{o, z_1, \dots, z_k\}$). We note that

$$\begin{aligned} g_k(z_1,\dots,z_k;\mathcal{Z}\cup\{z_1,\dots,z_k\}) \quad \leq \quad & g_{k-1}(z_1,\dots,z_{k-1};\mathcal{Z}\cup\{z_1,\dots,z_{k-1}\})\cdot \\ & g_{Y_{\mathbf{I}}}(z_1,\dots,z_k,\mathcal{Z}), \end{aligned} \tag{17}$$

where

$$g_{Y_{\mathbf{I}}}(z_1,\dots,z_k,\mathcal{Z}) = 1\left\{\begin{array}{l} r - w_1 < \text{dist}_{\mathbb{H}^2}(z_{k-1}, z_k) < r + w_2, \text{ and;} \\ \angle z_{n-2} z_{k-1} z_k > \vartheta, \text{ and;} \\ \text{either } \exists \text{ a disk } B \text{ with } z_{k-1}, z_k \in \partial B, \text{diam}_{\mathbb{H}^2}(B) < r + w_2, \\ \qquad \mathcal{Z}\cap B \setminus \text{cert}(o, z_1,\dots,z_{k-1}) = \emptyset, \\ \quad \text{or } \mathcal{Z}\cap DD^-(z_{k-1}, z_k, r+w_2)\setminus \text{cert}(o,z_1,\dots,z_{k-1}) = \emptyset, \\ \quad \text{or } \mathcal{Z}\cap DD^+(z_{k-1}, z_k, r+w_2)\setminus \text{cert}(o,z_1,\dots,z_{k-1}) = \emptyset, \\ \text{(or several of the above three hold.)} \end{array}\right\}.$$

Next, we note that if $o, z_1, \dots, z_k$ is not a good pre-pseudopath then $g_k(z_1,\dots,z_k,\mathcal{Z}\cup\{z_1,\dots,z_k\}) = 0$ and whenever $o, z_1, \dots, z_k$ is a good pre-pseudopath then

$$\begin{gathered} \mathbb{E}\left[g_k(z_1,\dots,z_{k-1};\mathcal{Z}\cup\{z_1,\dots,z_{k-1}\})\cdot g_{Y_{\mathbf{I}}}(z_1,\dots,z_k,\mathcal{Z})\right] \\ = \\ \mathbb{E}\left[g_{k-1}(z_1,\dots,z_{k-1};\mathcal{Z}\cup\{z_1,\dots,z_{k-1}\})\right]\cdot\mathbb{E}\left[g_{Y_{\mathbf{I}}}(z_1,\dots,z_k,\mathcal{Z})\right], \end{gathered} \tag{18}$$

as $\mathcal{Z}\cap\text{cert}(o,z_1,\dots,z_{k-1})$ and $\mathcal{Z}\setminus\text{cert}(o,z_1,\dots,z_{k-1})$ are independent; and the former determines the value of $g_k(z_1,\dots,z_{k-1};\mathcal{Z}\cup\{z_1,\dots,z_{k-1}\})$, while the latter determines $g_{Y_{\mathbf{I}}}(z_1,\dots,z_k,\mathcal{Z})$.

Next, notice that by Corollary 5 and Lemmas 44 and 45, we have

$$p\lambda \int_{\mathbb{D}} \mathbb{E}\left[g_{Y_{\mathbf{I}}}(z_1,\dots,z_k,\mathcal{Z})\right] f(z_k)\,\mathrm{d}\,z_k \le 1-\varepsilon/2+10^4 e^{w_2} e^{-e^{w_2/2}}, \tag{19}$$

for every $z_1,\dots,z_{k-1}$ such that $o,z_1,\dots,z_{k-1}$ is a good pre-pseudopath. If $o,z_1,\dots,z_{k-1}$ is not a good pre-pseudopath then $g_{k-1}(z_1,\dots,z_{k-1};\mathcal{Z}\cup\{z_1,\dots,z_{k-1}\}) =$ $g_k(z_1,\dots,z_n;\mathcal{Z}\cup\{z_1,\dots,z_n\}) = 0$. Combining (16)–(19), we see that

$$\begin{aligned}
\mathbb{E}G_k &\le (p\lambda)^n \int_{\mathbb{D}}\cdots\int_{\mathbb{D}} \mathbb{E}\left[g_{k-1}(z_1,\dots,z_{k-1};\mathcal{Z}\cup\{z_1,\dots,z_{k-1}\})\right]\cdot \\
&\qquad\qquad \mathbb{E}\left[g_{Y_{\mathbf{I}}}(z_1,\dots,z_k,\mathcal{Z})\right] f(z_1)\dots f(z_k)\,\mathrm{d}\,z_k\dots\mathrm{d}\,z_1 \\
&= (p\lambda)^{n-1} \int_{\mathbb{D}}\cdots\int_{\mathbb{D}} \mathbb{E}\left[g_{k-1}(z_1,\dots,z_{k-1};\mathcal{Z}\cup\{z_1,\dots,z_{k-1}\})\right]\cdot \\
&\qquad\qquad \left(p\lambda\int_{\mathbb{D}} \mathbb{E}\left[g_{Y_{\mathbf{I}}}(z_1,\dots,z_k,\mathcal{Z})\right] f(z_k)\,\mathrm{d}\,z_k\right) f(z_1)\dots f(z_{k-1})\,\mathrm{d}\,z_{k-1}\dots\mathrm{d}\,z_1 \\
&\le \left(1-\varepsilon/2+10^4 e^{w_2}e^{-e^{w_2/2}}\right)\cdot \\
&\qquad (p\lambda)^{n-1}\cdot\int_{\mathbb{D}}\cdots\int_{\mathbb{D}} \mathbb{E}\left[g_{k-1}(z_1,\dots,z_{k-1};\mathcal{Z}\cup\{z_1,\dots,z_{k-1}\})\right]\cdot \\
&\qquad\qquad f(z_1)\dots f(z_{k-1})\,\mathrm{d}\,z_{k-1}\dots\mathrm{d}\,z_1 \\
&= \left(1-\varepsilon/2+10^4 e^{w_2}e^{-e^{w_2/2}}\right)\cdot \mathbb{E}G_{k-1}.
\end{aligned}$$

The lemma follows by iterating the recursive inequality. ■

The previous lemma immediately gives that, provided we chose w_2 a sufficiently large constant (how large we need to take it depends on ε), for all sufficiently small intensities λ, almost surely, there are no infinite, black, good pseudopaths starting from the origin. In fact something slightly stronger holds:

Corollary 47 *For every $0<\varepsilon<1$ there exists a $c=c(\varepsilon)$ such that for all $w_1,\vartheta>0$ and $w_2>c$ the following holds.*

There exists a $\lambda_0=\lambda_0(\varepsilon,w_1,w_2,\vartheta)$ such that, when $0<\lambda<\lambda_0$ and $p\le(1-\varepsilon)\cdot(\pi/3)\cdot\lambda$, setting $r:=2\ln(1/\lambda)$, almost surely there are no infinite, black, good pseudopaths.

Proof. As pointed out earlier, we can choose $c=c(\varepsilon)$ such that for all $w_2>c$ and $w_1,\vartheta>0$ there is a λ_0 such that, whenever when $0<\lambda<\lambda_0$ and $p\le(1-\varepsilon)\cdot(\pi/3)\cdot\lambda$, almost surely there is no infinite, black, good pseudopath starting from the origin.

Let $x>0$ be arbitrary and write

$$N_x := \left|\left\{ z\in\mathcal{Z}_b\cap B_{\mathbb{H}^2}(o,x) : \begin{array}{l}\text{there is an infinite, black, good}\\ \text{pseudopath starting from } z\end{array}\right\}\right|.$$

By Corollary 5

$$\mathbb{E}N_x = p\lambda\int_{\mathbb{D}} \mathbb{E}\left[g(z,\mathcal{Z}\cup\{z\})\right] f(z)\,\mathrm{d}\,z,$$

where f is given by (2) as usual and $g(u,\mathcal{U})=1$ if $u\in B_{\mathbb{H}^2}(o,x)$ and $u\in\mathcal{U}$ and there is an infinite, black, good, pseudopath (wrt. r,w_1,w_2,ϑ and the point set $\mathcal{U}$) starting from u;

and $g(u,\mathcal{U}) = 0$ otherwise. If $z \notin B_{\mathbb{H}^2}(o,x)$ then $g(z, \mathcal{Z} \cup \{z\}) = 0$ by definition of g, and if $z \in B_{\mathbb{H}^2}(o,x)$ then

$$\mathbb{E}\left[g(z, \mathcal{Z} \cup \{z\})\right] = \mathbb{E}\left[g(o, \mathcal{Z} \cup \{o\})\right] = 0,$$

since the distribution of $\mathcal{Z}$ is invariant under the action of isometries of the Poincaré disk ($\mathcal{Z} \overset{d}{=} \varphi[\mathcal{Z}]$ if φ is an isometry), and in particular under an isometry that maps z to o; and we know that the origin a.s. is not on any infinite, back, good, pseudopath as remarked just before the statement of the lemma. It follows that $\mathbb{E}N_x = 0$ and hence also $N_x = 0$ a.s., for each $x > 0$ separately. Thus

$$\mathbb{P}\left(\exists \text{ infinite, black, good pseudopath}\right) = \mathbb{P}\left(\bigcup_{x>0} \{N_x \neq 0\}\right) = \lim_{x\to\infty} \mathbb{P}(N_x \neq 0) = 0,$$

as $\{N_x \neq 0\} \subseteq \{N_y \neq 0\}$ whenever $x < y$. ■

Recall that our strategy for the proof that $p_c \geq (1-\varepsilon)\cdot(\pi/3)\cdot\lambda$ for small λ is to prove that almost surely no black, infinite, good pseudopath and no black, infinite, linked sequence of chunks starting from the origin will exist (for an appropriate choice of $w_1, w_2, \vartheta > 0$ and $r = 2\ln(1/\lambda)$). In the light of the last corollary, we now turn our attention to linked sequences of chunks.

3.3.3 Counting chunks starting from the origin

Next, we bound the number of black chunks starting from the origin. We will exploit our results on good pseudopaths, using that a chunk is just a good pseudopath with a single bad pseudo-edge stuck to the end. We begin with a few lemmas designed for adding a last, bad edge to an existing good pseudopath. The names of the random variables described by the lemmas are L with some subscript, where L stands for "last" and the subscript corresponds to the type of edge under consideration.

Lemma 48 *For all $w_1, w_2, \vartheta > 0$ there exist $\lambda_0 = \lambda_0(w_1, w_2, \vartheta) > 0$ such that the following holds for all $0 < \lambda < \lambda_0$ and all $p \leq 10\lambda$, setting $r := 2\log(1/\lambda)$.*

Let $u_0, \dots, u_k \in \mathbb{D}$ be arbitrary, and set $C := \operatorname{cert}(u_0, \dots, u_k)$ and

$$L_{\textbf{IV-i},\geq d} := \left|\left\{ z \in \mathcal{Z}_b : \begin{array}{l} \operatorname{dist}_{\mathbb{H}^2}(u_k, z) \geq d, \text{ and;} \\ \operatorname{area}_{\mathbb{H}^2}(B_{Gab}(u_k,z) \cap C) \leq \frac{1}{10}\cdot \operatorname{area}_{\mathbb{H}^2}(B_{Gab}(u_k,z)), \text{ and;} \\ \mathcal{Z} \cap B^-_{Gab}(u_k,z) \setminus C = \emptyset \text{ or } \mathcal{Z} \cap B^+_{Gab}(u_k,z) \setminus C = \emptyset. \end{array} \right\}\right|.$$

We have

$$\mathbb{E}L_{\textbf{IV-i},\geq r+v} \leq 1000 \cdot e^{v/2} \cdot e^{-e^{v/2}},$$

for all $v \geq 0$.

Proof. We let $\lambda_0 > 0$ be a small constant to be determined in the course of the proof, and take an arbitrary $\lambda < \lambda_0$ and $p \leq 10\lambda$. Without loss of generality (applying a suitable isometry if needed) we can assume $u_k = o$ is the origin.

If z satisfies $\mathrm{dist}_{\mathbb{H}^2}(o,z) \geq r$ and $\mathrm{area}_{\mathbb{H}^2}(B_{\mathrm{Gab}}(o,z)\cap C) \leq \frac{1}{10}\cdot \mathrm{area}_{\mathbb{H}^2}(B_{\mathrm{Gab}}(o,z))$ then of course also

$$\mathrm{area}_{\mathbb{H}^2}(B^-_{\mathrm{Gab}}(o,z)\setminus C) \geq \left(\frac{1}{2}-\frac{1}{10}\right)\cdot \mathrm{area}_{\mathbb{H}^2}(B_{\mathrm{Gab}}(o,z)) \geq e^{\mathrm{dist}_{\mathbb{H}^2}(o,z)/2},$$

using that

$$\mathrm{area}_{\mathbb{H}^2}(B_{\mathrm{Gab}}(o,z)) = 2\pi\left(\cosh\left(\mathrm{dist}_{\mathbb{H}^2}(o,z)/2\right)-1\right) = (1+o_{\lambda_0}(1))\pi e^{\mathrm{dist}_{\mathbb{H}^2}(o,z)/2},$$

(the $o_{\lambda_0}(1)$ term referring to the situation where λ_0 tends to zero) and assuming that we have chosen λ_0 sufficiently small (so that $\mathrm{dist}_{\mathbb{H}^2}(o,z) \geq r \geq 2\ln(1/\lambda_0)$ is large). Analogously we also have

$$\mathrm{area}_{\mathbb{H}^2}(B^+_{\mathrm{Gab}}(o,z)\setminus C) \geq e^{\mathrm{dist}_{\mathbb{H}^2}(o,z)/2},$$

for all such z. By Corollary 5 and this last observation:

$$\begin{aligned}
\mathbb{E}L_{\textbf{IV-i},\geq r+v} &= p\lambda \int_{\mathbb{D}\setminus B_{\mathbb{H}^2}(o,r+v)} 1_{\{\mathrm{area}_{\mathbb{H}^2}(B_{\mathrm{Gab}}(o,z)\cap C)\leq \frac{1}{10}\cdot \mathrm{area}_{\mathbb{H}^2}(B_{\mathrm{Gab}}(o,z))\}}\cdot \\
&\qquad \mathbb{P}\left(\mathcal{Z}\cap B^-_{\mathrm{Gab}}(o,z)\setminus C=\emptyset \text{ or } \mathcal{Z}\cap B^+_{\mathrm{Gab}}(o,z)\setminus C=\emptyset\right) f(z)\,\mathrm{d}\,z \\
&\leq p\lambda \int_{\mathbb{D}\setminus B_{\mathbb{H}^2}(o,r+v)} 2e^{-\lambda e^{\mathrm{dist}_{\mathbb{H}^2}(o,z)/2}} f(z)\,\mathrm{d}\,z,
\end{aligned}$$

with f given by (2).

Applying a change to hyperbolic polar coordinates to z we find

$$\begin{aligned}
\mathbb{E}L_{\textbf{IV-i},\geq r+v} &\leq 2p\lambda \int_{r+v}^{\infty}\int_0^{2\pi} e^{-\lambda e^{\rho/2}} \sinh(\rho)\,\mathrm{d}\,\alpha\,\mathrm{d}\,\rho \\
&\leq 200\lambda^2 \int_{r+v}^{\infty} e^{-\lambda e^{\rho/2}} e^{\rho}\,\mathrm{d}\,\rho.
\end{aligned}$$

Applying the substitution $t=\lambda e^{\rho/2}$ (so that $\mathrm{d}\,\rho = \frac{2\,\mathrm{d}\,t}{t}$), we find

$$\begin{aligned}
\mathbb{E}L_{\textbf{IV-i},\geq r+v} &\leq 200\lambda^2 \int_{\lambda e^{(r+v)/2}}^{\infty} e^{-t}\cdot\left(\frac{t}{\lambda}\right)^2 \frac{2\,\mathrm{d}\,t}{t} \\
&= 400\int_{e^{v/2}}^{\infty} te^{-t}\,\mathrm{d}\,t \\
&= 400(e^{v/2}+1)e^{-e^{v/2}} \\
&\leq 1000\cdot e^{v/2}\cdot e^{-e^{v/2}}.
\end{aligned}$$

■

Corollary 49 *For all $w_1, w_2, \vartheta > 0$ there exist $\lambda_0 = \lambda_0(w_1,w_2,\vartheta) > 0$ such that the following holds for all $0<\lambda<\lambda_0$ and all $p \leq 10\lambda$, setting $r := 2\log(1/\lambda)$.*

Let $u_0,\dots,u_k \in \mathbb{D}$ be an arbitrary good pre-pseudopath, and set $C := \mathrm{cert}(u_0,\dots,u_k)$ and

$$L_{\mathbf{IV},\geq d} := \left|\left\{ z \in \mathcal{Z}_b : \begin{array}{l} \text{dist}_{\mathbb{H}^2}(u_k, z) \geq d, \ \textit{and;} \\ \mathcal{Z} \cap B^-_{Gab}(u_k, z) \setminus C = \emptyset \ \textit{or} \ \mathcal{Z} \cap B^+_{Gab}(u_k, z) \setminus C = \emptyset. \end{array} \right\}\right|.$$

We have

$$\mathbb{E}L_{\mathbf{IV},\geq r+v} \leq 10^4 \cdot e^{v/2} \cdot e^{-e^{v/2}},$$

for all $v \geq 0$.

Proof. We let $\lambda_0, \vartheta' > 0$ be small constants to be determined in the course of the proof, and take an arbitrary $0 < \lambda < \lambda_0$ and $p \leq 10\lambda$. Without loss of generality (applying a suitable isometry if needed) we can assume $u_k = o$ is the origin.

Since $v \geq 0$, by Corollary 38 with $\varepsilon := \frac{1}{10}$ and $\vartheta_1 = \vartheta, \vartheta_2 = \vartheta'$, assuming we have chosen λ_0 sufficiently small, we have

$$L_{\mathbf{IV},\geq r+v} \leq L_{\mathbf{IV\text{-}i},\geq r+v} + L_{\mathbf{IV\text{-}ii}},$$

where $L_{\mathbf{IV\text{-}i},\geq r+v}$ is as defined in Lemma 48 above and

$$L_{\mathbf{IV\text{-}ii}} := \left|\mathcal{Z}_b \cap B_{\mathbb{H}^2}(o, r+w) \cap \text{sect}\left(o, u_{k-1}, \vartheta'\right)\right|.$$

setting $w := w_2 + 22 \ln 10$.

In particular $L_{\mathbf{IV},\geq r+v} = L_{\mathbf{IV\text{-}i},\geq r+v}$ if $v \geq w$. In this case we are clearly done by Lemma 48.

To prove it also for $0 \leq v < w$, we also need to bound $\mathbb{E}L_{\mathbf{IV\text{-}ii}}$. Lemma 20 shows that

$$\mathbb{E}L_{\mathbf{IV\text{-}ii}} \leq 1000\vartheta' e^w \leq 1000 \cdot e^{v/2} \cdot e^{-e^{v/2}},$$

having chosen ϑ' sufficiently small for the second inequality. (To be precise, having chosen $\vartheta_2 < e^{-w} \cdot \min\limits_{0 \leq x \leq w} e^{x/2} \cdot e^{-e^{x/2}}$.)

Adding the bounds on $\mathbb{E}L_{\mathbf{IV\text{-}i},\geq r+v}$ and $\mathbb{E}L_{\mathbf{IV\text{-}ii}}$ proves the result for $0 \leq v \leq w$. ■

We will denote

$$C_k^{\mathbf{II}} := |\{\text{black chunks of length } k, \text{ starting from } o, \text{ with final pseudo-edge of type } \mathbf{II}\}|,$$

$$C_k^{\mathbf{III}} := |\{\text{black chunks of length } k, \text{ starting from } o, \text{ with final pseudo-edge of type } \mathbf{III}\}|,$$

and, for all $v \geq w_2$:

$$C_k^{\mathbf{IV},v} := \left|\left\{ \begin{array}{l} \text{black chunks of length } k, \text{ starting from } o, \text{ with final pseudo-edge of type } \mathbf{IV}, \\ \text{and the final pseudo-edge having length} \in [r+v, r+v+1) \end{array} \right\}\right|.$$

Recall that G_k denotes the number of black, good, pseudopaths starting from the origin of length k.

Corollary 50 *For every w_1, w_2, ϑ there exists a $\lambda_0 = \lambda_0(w_1, w_2, \vartheta) > 0$ such that the following holds for all $0 < \lambda < \lambda_0$ and $p \leq 10\lambda$, setting $r := 2\ln(1/\lambda)$.*

We have

$$\mathbb{E}C_k^{\mathbf{II}} \leq 10^3 \cdot e^{-w_1} \cdot \mathbb{E}G_{k-1}, \tag{20}$$

$$\mathbb{E}C_k^{\mathbf{III}} \leq 10^3 \cdot \vartheta \cdot e^{w_2} \cdot \mathbb{E}G_{k-1}, \tag{21}$$

and

$$\mathbb{E}C_k^{\mathbf{IV},v} \leq 10^4 \cdot e^{v/2} \cdot e^{-e^{v/2}} \cdot \mathbb{E}G_{k-1}, \tag{22}$$

for all $v \geq w_2$.

Proof. We will argue analogously to Lemma 46. We start by considering $C_k^{\mathbf{II}}$. By Corollary 5

$$\mathbb{E}C_k^{\mathbf{II}} = (p\lambda)^k \int_{\mathbb{D}} \cdots \int_{\mathbb{D}} \mathbb{E}\left[g_k^{\mathbf{II}}(z_1, \ldots, z_k, \mathcal{Z} \cup \{z_1, \ldots, z_k\})\right] \cdot f(z_1) \cdot \cdots \cdot f(z_k) \,\mathrm{d}\, z_k \ldots \mathrm{d}\, z_1,$$

where f is given by (2) and $g_k^{\mathbf{II}}$ is the indicator function that $o, z_1, \ldots, z_k$ is a chunk (wrt. the point set $\mathcal{Z} \cup \{o, z_1, \ldots, z_k\}$), whose last pseudo-edge $z_{k-1}z_k$ is of type **II**.

Let h_{k-1} denote the indicator function that $o, z_1, \ldots, z_{k-1}$ is a good pseudo-path wrt. the point set $\mathcal{Z} \cup \{o, z_1, \ldots, z_{k-1}\}$. We have

$$\begin{aligned} g_k^{\mathbf{II}}(z_1, \ldots, z_k, \mathcal{Z} \cup \{z_1, \ldots, z_k\}) \quad \leq \quad & h_{k-1}(z_1, \ldots, z_{k-1}, \mathcal{Z} \cup \{z_1, \ldots, z_{k-1}\}) \cdot \\ & 1_{\{\mathrm{dist}_{\mathbb{H}^2}(z_{k-1}, z_k) \leq r - w_1\}}. \end{aligned}$$

It follows that

$$\begin{aligned} \mathbb{E}C_k^{\mathbf{II}} \quad \leq \quad & (p\lambda)^{k-1} \int_{\mathbb{D}} \cdots \int_{\mathbb{D}} \mathbb{E}\left[h_{k-1}(z_1, \ldots, z_{k-1}, \mathcal{Z} \cup \{z_1, \ldots, z_{k-1}\})\right] \cdot f(z_1) \cdot \cdots \cdot f(z_{k-1}) \cdot \\ & \qquad \left(p\lambda \int_{\mathbb{D}} 1_{\{\mathrm{dist}_{\mathbb{H}^2}(z_{k-1}, z_k) \leq r - w_1\}} f(z_k) \,\mathrm{d}\, z_k\right) \mathrm{d}\, z_{k-1} \ldots \mathrm{d}\, z_1 \\ = \quad & p\lambda \cdot \mathrm{area}_{\mathbb{H}^2}(B_{\mathbb{H}^2}(o, r - w_1)) \cdot \\ & (p\lambda)^{k-1} \cdot \int_{\mathbb{D}} \cdots \int_{\mathbb{D}} \mathbb{E}\left[h_{k-1}(z_1, \ldots, z_{k-1}, \mathcal{Z} \cup \{z_1, \ldots, z_{k-1}\})\right] \cdot \\ & \qquad f(z_1) \cdot \cdots \cdot f(z_{k-1}) \,\mathrm{d}\, z_{k-1} \ldots \mathrm{d}\, z_1 \\ = \quad & p\lambda \, \mathrm{area}_{\mathbb{H}^2}(B_{\mathbb{H}^2}(o, r - w_1)) \cdot \mathbb{E}G_{k-1}, \end{aligned}$$

Applying Lemma 19 gives (20).

The proof of (21) is analogous, using the indicator function $1_{\{\mathrm{dist}_{\mathbb{H}^2}(z_{k-1}, z_k) < r + w_2, \angle z_{k-2} z_{k-1} z_k \leq \vartheta\}}$ in place of $1_{\{\mathrm{dist}_{\mathbb{H}^2}(z_{k-1}, z_k) \leq r - w_1\}}$, replacing $\mathrm{area}_{\mathbb{H}^2}(B_{\mathbb{H}^2}(o, r - w_1))$ with $\mathrm{area}_{\mathbb{H}^2}(B_{\mathbb{H}^2}(o, r + w_2) \cap \mathrm{sect}\,(o, v, \vartheta))$ (for $v \neq o$ arbitrary) and using Lemma 20 in place of Lemma 19.

For the proof of (22), we use that if $g_k^{\mathbf{IV},v}$ denotes the indicator function that $o, z_1, \ldots, z_k$ form a chunk (wrt. $\mathcal{Z} \cup \{o, z_1, \ldots, z_k\}$) whose last pseudo-edge has length $\in [r + v, r + v + 1)$ then

$$g_k^{\mathbf{IV},v}(z_1,\dots,z_k,\mathcal{Z}\cup\{z_1,\dots,z_k\}) \leq h_{k-1}(z_1,\dots,z_{k-1},\mathcal{Z}\cup\{z_1,\dots,z_{k-1}\})\cdot g_{L_{\mathbf{IV},\geq r+v}}(z_1,\dots,z_k,\mathcal{Z}),$$

where

$$g_{L_{\mathbf{IV},\geq r+v}}(z_1,\dots,z_k,\mathcal{Z}) := 1\left\{\begin{array}{l}\mathrm{dist}_{\mathbb{H}^2}(z_{k-1},z_k)\geq r+v,\ \text{and;}\\ \text{either } \mathcal{Z}\cap B^-_{\mathrm{Gab}}(z_{k-1},z_k)\setminus \mathrm{cert}(o,z_1,\dots,z_{k-1})=\emptyset,\\ \text{or } \mathcal{Z}\cap B^+_{\mathrm{Gab}}(z_{k-1},z_k)\setminus \mathrm{cert}(o,z_1,\dots,z_{k-1})=\emptyset,\\ \text{(or both).}\end{array}\right\}.$$

Since $\mathcal{Z}\cap\mathrm{cert}(o,z_1,\dots,z_{k-1})$ and $\mathcal{Z}\setminus\mathrm{cert}(o,z_1,\dots,z_{k-1})$ are independent for any choice of $z_1,\dots,z_{k-1}$, we have

$$\mathbb{E}\left[g_k^{\mathbf{IV},v}(z_1,\dots,z_k,\mathcal{Z}\cup\{z_1,\dots,z_k\})\right] \leq \mathbb{E}\left[h_{k-1}(z_1,\dots,z_{k-1},\mathcal{Z}\cup\{z_1,\dots,z_{k-1}\})\right]\cdot \mathbb{E}\left[g_{L_{\mathbf{IV},\geq r+v}}(z_1,\dots,z_k,\mathcal{Z})\right].$$

If $z_1,\dots,z_{k-1}$ do not form a good pre-pseudopath then $h_{k-1}(z_1,\dots,z_{k-1},\mathcal{Z}\cup\{z_1,\dots,z_{k-1}\})=0$. Otherwise we can apply Corollary 49 (and Corollary 5) to show that

$$p\lambda\int_{\mathbb{D}}\mathbb{E}\left[g_{L_{\mathbf{IV},\geq r+v}}(z_1,\dots,z_k,\mathcal{Z})\right]f(z_k)\,\mathrm{d}\,z_k\leq 10^4e^{v/2}e^{-e^{v/2}}.$$

In other words,

$$\begin{aligned}\mathbb{E}C_k^{\mathbf{IV},v} &= (p\lambda)^k\int_{\mathbb{D}}\cdots\int_{\mathbb{D}}\mathbb{E}\left[g_k^{\mathbf{IV},\geq r+v}(z_1,\dots,z_k,\mathcal{Z}\cup\{z_1,\dots,z_k\})\right]\cdot f(z_1)\cdot\dots\cdot f(z_k)\,\mathrm{d}\,z_k\dots\mathrm{d}\,z_1\\ &\leq (p\lambda)^{k-1}\int_{\mathbb{D}}\cdots\int_{\mathbb{D}}\mathbb{E}\left[h_{k-1}(z_1,\dots,z_{k-1},\mathcal{Z}\cup\{z_1,\dots,z_{k-1}\})\right]\cdot f(z_1)\cdot\dots\cdot f(z_{k-1})\cdot\\ &\qquad\left(p\lambda\int_{\mathbb{D}}\mathbb{E}\left[g_{L_{\mathbf{IV},\geq r+v}}(z_1,\dots,z_k,\mathcal{Z})\right]f(z_k)\,\mathrm{d}\,z_k\right)\mathrm{d}\,z_{k-1}\dots\mathrm{d}\,z_1\\ &\leq \mathbb{E}G_{k-1}\cdot 10^4e^{v/2}e^{-e^{v/2}},\end{aligned}$$

as desired. ■

3.3.4 Counting linked sequences of chunks

Next, we turn attention to counting the number of linked sequences of chunks. The first few lemmas are designed for dealing with the first pseudo-edge of a new chunk that will be linked to an existing chunk. The names of the random variables described in these lemmas are F with some subscript, where F stands for "first" and the subscript corresponds to the type of the new pseudo-edge to be added and the type of link under consideration. (The link types being **a**, **b**, **c** corresponding to the cases listed in part **(ii)** of Definition 30.)

Lemma 51 *For every $w_1, w_2, \vartheta > 0$ there exist $\lambda_0 = \lambda_0(w_1, w_2, \vartheta)$ such that the following holds for all $0 < \lambda < \lambda_0$ and $p \leq 10\lambda$, setting $r := 2\log(1/\lambda)$.*

Let $u_0, \dots, u_k \in \mathbb{D}$ be an arbitrary pre-chunk and define

$$F_{\textbf{I-b}} := \left| \left\{ (z_1, z_2) \in \mathcal{Z}_b \times \mathcal{Z}_b : \begin{array}{l} 0 < \operatorname{dist}_{\mathbb{H}^2}(z_1, z_2) < r + w_2, \text{ and;} \\ \operatorname{cert}(z_1, z_2) \cap \operatorname{cert}(u_0, \dots, u_k) \neq \emptyset. \end{array} \right\} \right|.$$

Then we have

$$\mathbb{E}F_{\textbf{I-b}} \leq 10^4 \cdot k \cdot \exp\left[\frac{3}{2}w_2 + \max\left(w_2, \operatorname{dist}_{\mathbb{H}^2}(u_{k-1}, u_k) - r\right)/2\right].$$

Proof. We can write $\operatorname{cert}(u_0, \dots, u_k) = B_1 \cup \dots \cup B_m$ with $m \leq 2k$ and $B_1 = B_{\mathbb{H}^2}(c_1, s_1), \dots, B_m = B_{\mathbb{H}^2}(c_m, s_m)$ balls whose radii are either $(r + w_2)/2$ or $\operatorname{dist}_{\mathbb{H}^2}(u_{k-1}, u_k)/2$.

For $i = 1, \dots, m$ we define

$$F_i := \left| \left\{ (z_1, z_2) \in \mathcal{Z}_b \times \mathcal{Z}_b : \begin{array}{l} 0 < \operatorname{dist}_{\mathbb{H}^2}(z_1, z_2) < r + w_2, \text{ and;} \\ B_{\mathbb{H}^2}(z_1, r + w) \cap B_i \neq \emptyset, \text{ and;} \\ B_{\mathbb{H}^2}(z_2, r + w) \cap B_i \neq \emptyset. \end{array} \right\} \right|.$$

Observe that if $\operatorname{dist}_{\mathbb{H}^2}(z_1, z_2) < r + w_2$ then $\operatorname{cert}(z_1, z_2)$ either equals $DD(z_1, z_2, r + w_2)$ or $\emptyset$. In particular we have $\operatorname{cert}(z_1, z_2) \subseteq B_{\mathbb{H}^2}(z_1, r + w) \cap B_{\mathbb{H}^2}(z_2, r + w)$.

We see that

$$F_{\textbf{I-b}} \leq F_1 + \dots + F_m.$$

We can thus focus on bounding the individual expectations $\mathbb{E}F_i$. For the moment, we pick some $1 \leq i \leq m$. Without loss of generality (applying a suitable isometry if needed) the center c_i of B_i is the origin o. Applying Corollary 5 and a (double) switch to hyperbolic polar coordinates:

$$\begin{aligned} \mathbb{E}F_i &= p^2\lambda^2 \int_0^\infty \int_0^\infty \int_0^{2\pi} \int_0^{2\pi} g(z_1(\alpha_1, \rho_1), z_2(\alpha_2, \rho_2)) \sinh(\rho_1) \sinh(\rho_2) \,\mathrm{d}\,\alpha_2 \,\mathrm{d}\,\alpha_1 \,\mathrm{d}\,\rho_2 \,\mathrm{d}\,\rho_1 \\ &\leq 100\lambda^4 \int_0^\infty \int_0^\infty \int_0^{2\pi} \int_0^{2\pi} g(z_1(\alpha_1, \rho_1), z_2(\alpha_2, \rho_2)) e^{\rho_1 + \rho_2} \,\mathrm{d}\,\alpha_2 \,\mathrm{d}\,\alpha_1 \,\mathrm{d}\,\rho_2 \,\mathrm{d}\,\rho_1, \end{aligned}$$

where

$$g(u_1, u_2) := 1_{\left\{ \begin{array}{l} 0 < \operatorname{dist}_{\mathbb{H}^2}(u_1, u_2) < r + w_2, \text{ and;} \\ B_{\mathbb{H}^2}(u_1, r + w_2) \cap B_i \neq \emptyset, \text{ and;} \\ B_{\mathbb{H}^2}(u_2, r + w_2) \cap B_i \neq \emptyset. \end{array} \right\}}.$$

Note that $B_{\mathbb{H}^2}(z_i, r + w_2) \cap B_i \neq \emptyset$ implies that $\rho_i = \operatorname{dist}_{\mathbb{H}^2}(o, z_i) < s_i + r + w_2$. Applying Lemma 43, we see that $\operatorname{dist}_{\mathbb{H}^2}(z_1, z_1) < r + w_2$ implies that $\angle z_1 o z_2 = |\alpha_1 - \alpha_2|_{2\pi} < 2\pi e^{(r + w_2 - (\rho_1 + \rho_2))/2}$. (Here and elsewhere $|x|_{2\pi} := \min(|x|, 2\pi - |x|)$ for $-2\pi \leq x \leq 2\pi$.) Therefore

$$g(z_1, z_2) \leq 1_{\left\{ \begin{array}{l} \rho_1, \rho_2 < s_i + r + w_2, \\ |\alpha_1 - \alpha_2|_{2\pi} < 2\pi e^{(r + w_2 - (\rho_1 + \rho_2))/2} \end{array} \right\}}.$$

It follows that

$$\mathbb{E}F_i \leq 100\lambda^4 \int_0^{s_i+r+w_2}\int_0^{s_i+r+w_2}\int_0^{2\pi}\int_0^{2\pi} 1_{\{|\alpha_1-\alpha_2|_{2\pi}<2\pi e^{(r+w_2-\rho_1-\rho_2)/2}\}} \cdot e^{\rho_1+\rho_2}\,\mathrm{d}\,\alpha_2\,\mathrm{d}\,\alpha_1\,\mathrm{d}\,\rho_2\,\mathrm{d}\,\rho_1. \tag{23}$$

By symmetry considerations

$$\int_0^{2\pi}\int_0^{2\pi} 1_{\{|\alpha_1-\alpha_2|_{2\pi}<2\pi e^{(r+w_2-(\rho_1+\rho_2))/2}\}}\,\mathrm{d}\,\alpha_2\,\mathrm{d}\,\alpha_1 \leq 8\pi^2\cdot e^{(r+w_2-(\rho_1+\rho_2))/2}.$$

(We have an inequality and not an equality to account for the possibility that $2\pi e^{r+w_2-(\rho_1+\rho_2)} \geq \pi$.) Filling this back into (23), we obtain

$$\begin{aligned}
\mathbb{E}F_i &\leq 10^4\lambda^4 e^{(r+w_2)/2}\int_0^{s_i+r+w_2}\int_0^{s_i+r+w_2} e^{(\rho_1+\rho_2)/2}\,\mathrm{d}\,\rho_2\rho_1 \\
&= \tfrac{1}{4}\cdot 10^4\lambda^4 e^{(r+w_2)/2}\cdot e^{s_i+r+w_2} \\
&= \tfrac{1}{4}\cdot 10^4\cdot\exp\left[\tfrac{3}{2}w_2+s_i-r/2\right] \\
&\leq \tfrac{1}{4}\cdot 10^4\cdot\exp\left[\tfrac{3}{2}w_2+\max\left(w_2,\mathrm{dist}_{\mathbb{H}^2}(u_{k-1},u_k)-r\right)/2\right].
\end{aligned}$$

using in the last line that $2s_i$, the diameter of B_i, is either $r+w_2$ or $\mathrm{dist}_{\mathbb{H}^2}(u_{k-1},u_k)$. Adding the bounds on $\mathbb{E}Y_i'$ and using the inequality $m \leq 2k$ gives

$$\begin{aligned}
\mathbb{E}F_{\textbf{I-b}} &\leq \mathbb{E}F_1+\cdots+\mathbb{E}F_m \\
&\leq 2k\cdot\tfrac{1}{4}\cdot 10^4\cdot\exp\left[\tfrac{3}{2}w_2+\max\left(w_2,\mathrm{dist}_{\mathbb{H}^2}(u_{k-1},u_k)-r\right)/2\right] \\
&\leq 10^4\cdot k\cdot\exp\left[\tfrac{3}{2}w_2+\max\left(w_2,\mathrm{dist}_{\mathbb{H}^2}(u_{k-1},u_k)-r\right)/2\right],
\end{aligned}$$

as claimed in the lemma statement. ■

Lemma 52 *For every $w_1, w_2, \vartheta > 0$ there exist $\lambda_0 = \lambda_0(w_1, w_2, \vartheta)$ such that the following holds for all $0 < \lambda < \lambda_0$ and $p \leq 10\lambda$, setting $r := 2\log(1/\lambda)$.*

Let $u_0, \dots, u_k \in \mathbb{D}$ be an arbitrary pre-chunk and define

$$F_{\textbf{I-c}} := \left|\left\{(z_1,z_2)\in\mathcal{Z}_b\times\mathcal{Z}_b : \begin{array}{l} 0<\mathrm{dist}_{\mathbb{H}^2}(z_1,z_2)<r+w_2,\ \textit{and;} \\ z_1z_2\ \textit{is a pseudoedge, and;} \\ \mathrm{cert}(z_1,z_2)\cap\mathrm{cert}(u_0,\dots,u_k)=\emptyset,\ \textit{and;} \\ \mathrm{dist}_{\mathbb{H}^2}(z_1,\mathrm{cert}(u_0,\dots,u_k))<r/1000. \end{array}\right\}\right|.$$

Then we have

$$\mathbb{E}F_{\textbf{I-c}} \leq k\cdot\exp\left[\max\left(w_2,\mathrm{dist}_{\mathbb{H}^2}(u_{k-1},u_k)-r\right)/2\right].$$

Proof. As usual, we let $\lambda_0 > 0$ be a small constant, to be determined in the course of the proof. As in the previous proof, we write $C := \mathrm{cert}(u_0,\dots,u_k) = B_1\cup\cdots\cup B_m$ with $m \leq 2k$ and $B_1 = B_{\mathbb{H}^2}(c_1,s_1),\dots,B_m = B_{\mathbb{H}^2}(c_m,s_m)$ balls whose radii are either $(r+w_2)/2$ or $\mathrm{dist}_{\mathbb{H}^2}(u_{k-1},u_k)/2$. Let us write $C' := \{u\in\mathbb{D} : \mathrm{dist}_{\mathbb{H}^2}(u,C) < r/1000\}$. By Corollary 5

$$\begin{aligned}
\mathbb{E}F_{\textbf{I-c}} &\leq (p\lambda)^2 \int_{\mathbb{D}}\int_{\mathbb{D}} 1_{\{z_1\in C'\}} 1_{\left\{\begin{array}{l} z_1z_2 \text{ pseudoedge,} \\ \text{dist}_{\mathbb{H}^2}(z_1,z_2) < r+w_2 \end{array}\right\}} f(z_1)f(z_2)\,\mathrm{d}\,z_1\,\mathrm{d}\,z_2 \\
&= (p\lambda)^2 \,\text{area}_{\mathbb{H}^2}(C')\cdot \int_{B_{\mathbb{H}^2}(o,r+w_2)} 1_{\{0z \text{ pseudoedge}\}} f(z)\,\mathrm{d}\,z \\
&\leq p\lambda\,\text{area}_{\mathbb{H}^2}(C')\cdot \left(p\mathbb{E}D + \mathbb{E}X_{\mathbf{I}} + \mathbb{E}\tilde{X}_{\mathbf{VI}}\right) \\
&\leq 10\cdot\lambda^2\cdot 2k\cdot \pi e^{\max(r+w_2,\text{dist}_{\mathbb{H}^2}(u_{k-1},u_k))/2+r/1000}\cdot \\
&\qquad \left(10\lambda\cdot\left(6+\frac{3}{\pi\lambda}\right) + 1000e^{-w_2} + 1000e^{w_2}e^{-e^{w_2/2}}\right) \\
&= k\cdot e^{\max\left(w_2,\text{dist}_{\mathbb{H}^2}(u_{k-1},u_k)-r\right)/2}\cdot e^{-\left(\frac{499}{1000}\right)r}\cdot \\
&\qquad \cdot\left(120\pi\lambda + 60 + 2000\pi e^{-w_2} + 2000\pi e^{w_2}e^{-e^{w_2/2}}\right) \\
&\leq k\cdot e^{\max\left(w_2,\text{dist}_{\mathbb{H}^2}(u_{k-1},u_k)-r\right)/2},
\end{aligned}$$

where D is the "typical degree" as given by (8), $X_{\mathbf{I}}$ is as defined in Lemma 19 and $\tilde{X}_{\mathbf{VI}}$ as defined in Lemma 26, and; we apply Isokawa's formula (Theorem 6) and Lemmas 19 and 26 and we use that $p \leq 10\lambda$ and that $\text{area}_{\mathbb{H}^2}(B_{\mathbb{H}^2}(u,s)) \leq \pi e^s$ for all $u \in \mathbb{D}, s > 0$ to obtain the fourth line. In the fifth line we use that $r = 2\ln(1/\lambda)$ and in the last line that we chose $\lambda_0 = \lambda_0(w_1, w_2, \vartheta)$ sufficiently small (so that r is large). ■

Lemma 53 *For every $w_1, w_2, \vartheta > 0$ there exist $\lambda_0 = \lambda_0(w_1, w_2, \vartheta)$ such that the following holds for all $0 < \lambda < \lambda_0$ and $p \leq 10\lambda$, setting $r := 2\log(1/\lambda)$.*

Let $u_0, \dots, u_k \in \mathbb{D}$ be an arbitrary pre-chunk and write $C := \text{cert}(u_0, \dots, u_k)$ and

$$F_{\textbf{IV-a},\geq d} := \left|\left\{ z \in \mathcal{Z}_b : \begin{array}{l} \text{dist}_{\mathbb{H}^2}(z,u_k) \geq d, \text{ and;} \\ \text{dist}_{\mathbb{H}^2}(z,C) \geq r/1000, \text{ and;} \\ \mathcal{Z}\cap B^-_{Gab}(z,u_k)\setminus C = \emptyset \text{ or } \mathcal{Z}\cap B^+_{Gab}(z,u_k)\setminus C = \emptyset. \end{array}\right\}\right|.$$

Then we have

$$\mathbb{E}F_{\textbf{IV-a},\geq r+v} \leq 10^3\cdot e^{v/2}\cdot e^{-e^{v/2}},$$

for all $v \geq w_2$.

Proof. The demand that $\text{dist}_{\mathbb{H}^2}(z,C) > r/1000$ implies that

$$\text{ahd}(C, B_{\text{Gab}}(u_k,z)) > r/1000,$$

as well. Provided λ_0 is sufficiently small (so that r is sufficiently large), Lemma 41 (with $\varepsilon = 1/10$) implies that

$$\text{area}_{\mathbb{H}^2}(B_{\text{Gab}}(u_k,z)\cap C) \leq \frac{1}{10}\cdot \text{area}_{\mathbb{H}^2}(B_{\text{Gab}}(u_k,z)).$$

The result now follows from immediately Lemma 48. ■

Lemma 54 *For every $w_1, w_2, \vartheta > 0$ there exist $\lambda_0 = \lambda_0(w_1, w_2, \vartheta)$ such that the following holds for all $0 < \lambda < \lambda_0$ and $p \leq 10\lambda$, setting $r := 2\log(1/\lambda)$.*

Let $u_0, \dots, u_k \in \mathbb{D}$ be an arbitrary pre-chunk and write $C := \mathrm{cert}(u_0, \dots, u_k)$ and

$$F_{\textbf{IV-b},\geq d} := \left| \left\{ (z_1, z_2) \in \mathcal{Z}_b \times \mathcal{Z}_b : \begin{array}{l} \mathrm{dist}_{\mathbb{H}^2}(z_1, z_2) \geq d, \text{ and;} \\ \mathrm{dist}_{\mathbb{H}^2}(z_1, C) \geq r/1000, \text{ and;} \\ B_{Gab}(z_1, z_2) \cap C \neq \emptyset, \text{ and;} \\ \mathcal{Z} \cap B^-_{Gab}(z_1, z_2) \setminus C = \emptyset \text{ or } \mathcal{Z} \cap B^+_{Gab}(z_1, z_2) \setminus C = \emptyset. \end{array} \right\} \right|.$$

Then we have

$$\mathbb{E}F_{\textbf{IV-b},\geq r+v} \leq 10^6 \cdot k \cdot e^v \cdot e^{-e^{v/2}} \cdot e^{\max(w_2, \mathrm{dist}_{\mathbb{H}^2}(u_{k-1}, u_k) - r)/2},$$

for all $v \geq w_2$.

Proof. As usual, we let $\lambda_0 > 0$ be a small positive constant, to be determined during the course of the proof, and we fix an arbitrary $v \geq w_2$, $0 < \lambda < \lambda_0$ and $p \leq 10\lambda$.

We can write $\mathrm{cert}(u_0, \dots, u_k) = B_1 \cup \dots \cup B_m$ with $m \leq 2k$ and $B_1 = B_{\mathbb{H}^2}(c_1, s_1), \dots, B_m = B_{\mathbb{H}^2}(c_m, s_m)$ balls whose radii are either $(r + w_2)/2$ or $\mathrm{dist}_{\mathbb{H}^2}(u_{k-1}, u_k)/2$.

For $i = 1, \dots, m$ we define

$$F_i := \left| \left\{ (z_1, z_2) \in \mathcal{Z}_b \times \mathcal{Z}_b : \begin{array}{l} \mathrm{dist}_{\mathbb{H}^2}(z_1, z_2) \geq r + v, \text{ and;} \\ \mathrm{dist}_{\mathbb{H}^2}(z_1, C) > r/1000, \text{ and;} \\ B_{\mathrm{Gab}}(z_1, z_2) \cap B_i \neq \emptyset, \text{ and;} \\ \mathcal{Z} \cap B^-_{\mathrm{Gab}}(z_1, z_2) \setminus C = \emptyset \text{ or } \mathcal{Z} \cap B^+_{\mathrm{Gab}}(z_1, z_2) \setminus C \neq \emptyset. \end{array} \right\} \right|,$$

and we point out that

$$F_{\textbf{IV-b},\geq r+v} \leq F_1 + \dots + F_m.$$

In particular it suffices to bound each expectation $\mathbb{E}F_i$ separately. For the moment we fix $1 \leq i \leq m$. Applying a suitable isometry if needed, we assume without loss of generality that the center of B_i is $c_i = o$ the origin.

By Corollary 5:

$$\mathbb{E}F_i = p^2\lambda^2 \int_{\mathbb{D}} \int_{\mathbb{D}} \mathbb{E}\left[g(z_1, z_2, \mathcal{Z} \cup \{z_1, z_2\})\right] f(z_1) f(z_2) \,\mathrm{d}\, z_2 \,\mathrm{d}\, z_1,$$

with f given by (2) and

$$g(u_1, u_1, \mathcal{U}) := 1 \left\{ \begin{array}{l} \mathrm{dist}_{\mathbb{H}^2}(u_1, C) > r/1000, \text{ and;} \\ \mathrm{dist}_{\mathbb{H}^2}(u_1, u_2) \geq r + v, \text{ and;} \\ B_{\mathrm{Gab}}(u_1, u_2) \cap B_i \neq \emptyset, \text{ and;} \\ \mathcal{U} \cap B^-_{\mathrm{Gab}}(u_1, u_2) \setminus C = \emptyset \text{ or } \mathcal{U} \cap B^+_{\mathrm{Gab}}(z_1, z_2) \setminus C \neq \emptyset. \end{array} \right\}.$$

Applying a change to hyperbolic polar coordinates to z_1 we find

$$\begin{aligned} \mathbb{E}F_i = p^2\lambda^2 \int_{s+r/1000}^{\infty} \int_0^{2\pi} \int_{\mathbb{D}} \mathbb{E}\left[g(z_1(\alpha_1, \rho_1), z_2, \mathcal{Z} \cup \{z_1(\alpha_1, \rho_1), z_2\})\right] \cdot \\ f(z_2) \,\mathrm{d}\, z_2 \,\mathrm{d}\, \alpha_1 \sinh(\rho_1) \,\mathrm{d}\, \rho_1. \end{aligned} \tag{24}$$

Next, we consider the inner integral for some (fixed for the moment) $z_1 = (\cos(\alpha) \cdot \tanh(\rho_1/2), \sin(\alpha) \cdot \tanh(\rho_1/2))$ satisfying $\text{dist}_{\mathbb{H}^2}(z_1, C) > r/1000$. Let $\varphi : \mathbb{D} \to \mathbb{D}$ be a hyperbolic isometry that maps z_1 to the origin and $c_i = o$ to the negative x-axis. (So that $\varphi(c_i) = (-\tanh(r_1/2), 0)$.) Note that

$$\mathbb{E}\left[g(z_1, z_2, \mathcal{Z} \cup \{z_1, z_2\})\right] = h(\varphi(z_2)),$$

where

$$h(z) := 1_{\left\{\begin{array}{l} B_{\mathbb{H}^2}(\varphi(c_i), s_i) \cap B_{\text{Gab}}(o, z) \neq \emptyset, \\ \text{dist}_{\mathbb{H}^2}(o, z) > r + v, \\ \text{dist}_{\mathbb{H}^2}(o, C) \geq r/1000 \end{array}\right\}} \cdot \mathbb{P}\left(\begin{array}{l} \text{either } \mathcal{Z} \cap B^-_{\text{Gab}}(o, z) \setminus \varphi[C] = \emptyset, \\ \text{or } \mathcal{Z} \cap B^+_{\text{Gab}}(o, z) \setminus \varphi[C] = \emptyset, \\ \text{(or both).} \end{array}\right).$$

We point out that if $B_{\mathbb{H}^2}(\varphi(c), s_i) \cap B_{\text{Gab}}(o, z) \neq \emptyset$ then it must certainly hold that $\text{dist}_{\mathbb{H}^2}(o, z) > \text{dist}_{\mathbb{H}^2}(o, \varphi(c)) - s_i = \rho_1 - s$. By Lemma 42, using that $\text{dist}_{\mathbb{H}^2}(o, \varphi(c)) = \text{dist}_{\mathbb{H}^2}(z_1, c) = \rho_1 > s_i + r/1000$ by assumption, we must also have that

$$\angle \varphi(c) o z < 10 e^{(s_i - \rho_1)/2}.$$

We see that

$$1_{\left\{\begin{array}{l} B_{\mathbb{H}^2}(\varphi(c), s_i) \cap B_{\text{Gab}}(o, z) \neq \emptyset, \\ \text{dist}_{\mathbb{H}^2}(o, z) > r + v. \end{array}\right\}} \leq 1_{\left\{\begin{array}{l} \text{dist}_{\mathbb{H}^2}(o, z) > \max(\rho_1 - s_i, r + v), \\ \angle \varphi(c) o z < 10 e^{(s_i - \rho_1)/2} \end{array}\right\}}.$$

Since $\text{dist}_{\mathbb{H}^2}(o, \varphi[C]) = \text{dist}_{\mathbb{H}^2}(z_1, C) > r/1000$, we certainly have

$$\text{ahd}(\varphi[C], B_{\text{Gab}}(0, z)) > r/1000.$$

Assuming λ_0 was chosen sufficiently small, we can apply Lemma 41 to see that

$$\text{area}_{\mathbb{H}^2}(B_{\text{Gab}}(o, z) \cap \varphi[C]) \leq \frac{1}{1000} \cdot \text{area}_{\mathbb{H}^2}(B_{\text{Gab}}(o, z)),$$

and hence

$$\text{area}_{\mathbb{H}^2}(B^-_{\text{Gab}}(o, z) \setminus \varphi[C]) \geq \frac{499}{1000} \cdot \text{area}_{\mathbb{H}^2}(B_{\text{Gab}}(0, z)) \geq e^{\text{dist}_{\mathbb{H}^2}(o,z)/2},$$

where the last inequality holds assuming we chose λ_0 sufficiently small, and assuming that $\text{dist}_{\mathbb{H}^2}(o, z) > r + v$ (and using $r = 2\ln(1/\lambda)$). Completely analogously, $\text{area}_{\mathbb{H}^2}(B^+_{\text{Gab}}(o, z) \setminus \varphi[C]) \geq e^{\text{dist}_{\mathbb{H}^2}(o,z)/2}$ as well.

We see that

$$h(z) \leq 1_{\left\{\begin{array}{l} \text{dist}_{\mathbb{H}^2}(o, z) > \max(\rho_1 - s_i, r + v), \\ \angle \varphi(c) o z < 10 e^{(s_i - \rho_1)/2} \end{array}\right\}} \cdot 2 e^{-\lambda e^{\text{dist}_{\mathbb{H}^2}(o,z)/2}} =: \psi(z).$$

Applying (7), we find

$$\int_{\mathbb{D}} \mathbb{E}\left[g(z_1(\alpha, r_1), z_2, \mathcal{Z})\right] f(z_2) \,\mathrm{d}\, z_2 \leq \int_{\mathbb{D}} \psi(\varphi(z_2)) f(z_2) \,\mathrm{d}\, z_2 = \int_{\mathbb{D}} \psi(u) f(u) \,\mathrm{d}\, u.$$

Changing to hyperbolic coordinates, i.e. $u(\alpha_2, \rho_2) = (\cos(\alpha_2) \cdot \tanh(\rho_2/2), \sin(\alpha_2) \cdot \tanh(\rho_2/2))$ gives

$$\begin{aligned}
\int_{\mathbb{D}} \psi(u) f(u) \,\mathrm{d}\, u &= \int_0^\infty \int_0^{2\pi} \psi(u(\alpha_2, \rho_2)) \,\mathrm{d}\, \alpha_2 \sinh(\rho_2) \,\mathrm{d}\, \rho_2 \\
&\leq \int_{\max(\rho_1 - s_i, r+v)}^\infty \int_0^{2\pi} 1_{\{|\alpha_2 - \pi| < 10 e^{(s_i - \rho_1)/2}\}} 2 e^{-\lambda e^{\rho_2/2}} \,\mathrm{d}\, \alpha_2 \sinh(\rho_2) \,\mathrm{d}\, \rho_2 \\
&= \int_{\max(\rho_1 - s_i, r+v)}^\infty 20 e^{(s_i - \rho_1)/2} 2 e^{-\lambda e^{\rho_2/2}} \sinh(\rho_2) \,\mathrm{d}\, \rho_2 \\
&= 40 e^{(s - \rho_1)/2} \int_{\max(\rho_1 - s_i, r+v)}^\infty e^{-\lambda e^{\rho_2/2}} \sinh(\rho_2) \,\mathrm{d}\, \rho_2 \\
&\leq 20 e^{(s_i - \rho_1)/2} \int_{\max(\rho_1 - s_i, r+v)}^\infty e^{-\lambda e^{\rho_2/2}} e^{\rho_2} \,\mathrm{d}\, \rho_2
\end{aligned}$$

We next apply the substitution $t := \lambda e^{\rho_2/2}$ (so that $\mathrm{d}\, \rho_2 = \frac{2 \,\mathrm{d}\, t}{t}$) to obtain

$$\begin{aligned}
\int_{\mathbb{D}} \psi(u) f(u) \,\mathrm{d}\, u &\leq \frac{20 e^{(s_i - \rho_1)/2}}{\lambda^2} \cdot \int_{\lambda e^{\max(\rho_1 - s_i, r+v)/2}}^\infty t e^{-t} \,\mathrm{d}\, t \\
&= \frac{20 e^{(s_i - \rho_1)/2}}{\lambda^2} \cdot \left(\lambda e^{\max(\rho_1 - s_i, r+v)/2} + 1\right) \cdot e^{-\lambda e^{\max(\rho_1 - s_i, r+v)/2}} \\
&\leq \frac{40 e^{(s_i - \rho_1)/2}}{\lambda^2} \lambda e^{\max(\rho_1 - s_i, r+v)/2} e^{-\lambda e^{\max(\rho_1 - s_i, r+v)/2}},
\end{aligned}$$

using in the last line that $\lambda e^{r/2} = 1$ and $v \geq w > 0$.

Filling this back into (24) we find

$$\begin{aligned}
\mathbb{E} F_i &\leq p^2 \lambda^2 \int_{s_i + r/1000}^\infty \int_0^{2\pi} \frac{40 \pi e^{(s_i - \rho_1)/2}}{\lambda^2} \lambda e^{\max(\rho_1 - s_i, r+v)/2} e^{-\lambda e^{\max(\rho_1 - s_i, r+v)/2}} \,\mathrm{d}\, \alpha_1 \sinh(\rho_1) \,\mathrm{d}\, \rho_1 \\
&= 80 \pi p^2 \lambda e^{s_i/2} \int_{s + r/1000}^\infty e^{-\rho_1/2} \cdot e^{\max(\rho_1 - s_i, r+v)/2} e^{-\lambda e^{\max(\rho_1 - s_i, r+v)/2}} \sinh(\rho_1) \,\mathrm{d}\, \rho_1 \\
&\leq 10^4 \lambda^3 e^{s_i/2} \int_{s_i + r/1000}^\infty e^{\rho_1/2} \cdot e^{\max(\rho_1 - s_i, r+v)/2} e^{-\lambda e^{\max(\rho_1 - s_i, r+v)/2}} \,\mathrm{d}\, \rho_1 \\
&= 10^4 \lambda^3 e^{s_i/2} \cdot \left(\int_{s + r/1000}^{s_i + r + v} e^{\rho_1/2} \cdot e^{(r+v)/2} e^{-\lambda e^{(r+v)/2}} \,\mathrm{d}\, \rho_1 \right. \\
&\qquad \left. + \int_{s_i + r + v}^\infty e^{\rho_1/2} \cdot e^{(\rho_1 - s_i)/2} e^{-\lambda e^{(\rho_1 - s_i)/2}} \,\mathrm{d}\, \rho_1 \right) \\
&=: 10^4 \lambda^3 e^{s_i/2} \cdot (I_1 + I_2),
\end{aligned}$$

using that $\sinh(\rho_1) \leq \frac{1}{2} e^{\rho_1}$ and $p \leq 10\lambda$ in the third line.

Now

$$
\begin{aligned}
10^4\lambda^3 e^{s_i/2} I_1 &\leq 2\cdot 10^4\lambda^3 e^{s_i/2}\cdot e^{(s_i+r+v)/2}\cdot e^{(r+v)/2} e^{-\lambda e^{(r+v)/2}} \\
&= 2\cdot 10^4\cdot e^{s_i-r/2}\cdot e^{v}\cdot e^{-e^{v/2}},
\end{aligned} \tag{25}
$$

using that $r = 2\ln(1/\lambda)$.

We also have

$$
\begin{aligned}
10^4\lambda^3 e^{s_i/2} I_2 &= 10^4\lambda^3 e^{s_i}\int_{s_i+r+v}^{\infty} e^{\rho_1-s_i} e^{-\lambda e^{(\rho_1-s_i)/2}}\,\mathrm{d}\,\rho_1 \\
&= 10^4\lambda^3 e^{s_i}\int_{r+v}^{\infty} e^{u} e^{-\lambda e^{u/2}}\,\mathrm{d}\,u \\
&= 2\cdot 10^4\lambda e^{s_i}\int_{e^{v/2}}^{\infty} t e^{-t}\,\mathrm{d}\,t \\
&\leq 4\cdot 10^4\lambda e^{s_i} e^{v/2} e^{-e^{v/2}} \\
&\leq 4\cdot 10^4\cdot e^{s_i-r/2}\cdot e^{v}\cdot e^{-e^{v/2}},
\end{aligned} \tag{26}
$$

using the substitution $u = \rho_1 - s_i$ in the second line, the substitution $t := \lambda e^{u/2}$ in the third line and that $r = 2\ln(1/\lambda)$. Adding (25) and (26) and multiplying by $2k$ gives the result. ■

Lemma 55 *For every $w_1, w_2, \vartheta > 0$ there exist $\lambda_0 = \lambda_0(w_1, w_2, \vartheta)$ such that the following holds for all $0 < \lambda < \lambda_0$ and $p \leq 10\lambda$, setting $r := 2\log(1/\lambda)$.*

Let $u_0, \dots, u_k \in \mathbb{D}$ be an arbitrary pre-chunk and write $C := \mathrm{cert}(u_0, \dots, u_k)$ and

$$
F_{\textbf{IV-c},\geq d} := \left|\left\{ (z_1,z_2)\in\mathcal{Z}_b\times\mathcal{Z}_b : \begin{array}{l} \mathrm{dist}_{\mathbb{H}^2}(z_1,z_2)\geq d, \text{ and;} \\ \mathrm{dist}_{\mathbb{H}^2}(z_1,C)\leq r/1000, \text{ and;} \\ B_{Gab}(z_1,z_2)\cap C=\emptyset, \text{ and;} \\ \mathcal{Z}\cap B^-_{Gab}(z_1,z_2)=\emptyset \text{ or } \mathcal{Z}\cap B^+_{Gab}(z_1,z_2)=\emptyset. \end{array} \right\}\right|.
$$

Then we have

$$
\mathbb{E}F_{\textbf{IV-c},\geq r+v} \leq 10^3\cdot k\cdot e^{v}\cdot e^{-e^{v/2}}\cdot e^{\max(w_2,\mathrm{dist}_{\mathbb{H}^2}(u_{k-1},u_k)-r)/2},
$$

for all $v \geq w_2$.

Proof. As usual, we let $\lambda_0 > 0$ be a small constant, to be chosen appropriately during the course of the proof. Writing $C' := \{u \in \mathbb{D} : \mathrm{dist}_{\mathbb{H}^2}(u, C) < r/1000\}$ and applying Corollary 5, we have

$$
\begin{aligned}
&\mathbb{E}F_{\textbf{IV-b},\geq r+v} \\
&\leq \\
&(p\lambda)^2\int_{\mathbb{D}}\int_{\mathbb{D}} 1_{\{z_1\in C',\mathrm{dist}_{\mathbb{H}^2}(z_1,z_2)\geq r+v\}}\cdot\mathbb{P}(\mathcal{Z}\cap B^-_{\mathrm{Gab}}(z_1,z_2)=\emptyset \text{ or } \mathcal{Z}\cap B^+_{\mathrm{Gab}}(z_1,z_2)=\emptyset) \\
&f(z_1)f(z_2)\,\mathrm{d}\,z_2\,\mathrm{d}\,z_1 \\
&\leq \\
&p\lambda\cdot\mathrm{area}_{\mathbb{H}^2}(C')\cdot 10^3\cdot e^{v}\cdot e^{-e^{v/2}},
\end{aligned}
$$

applying Lemma 24. Next we remark

$$\begin{array}{rcl} p\lambda \cdot \mathrm{area}_{\mathbb{H}^2}(C') & \leq & 10\lambda^2 \cdot 2k \cdot \pi e^{\max(r+w_2, \mathrm{dist}_{\mathbb{H}^2}(u_{k-1},u_k))/2+r/1000} \\ & \leq & 20\pi \cdot k \cdot e^{\max(w_2, \mathrm{dist}_{\mathbb{H}^2}(u_{k-1},u_k)-r)/2-\left(\frac{499}{1000}\right)r} \\ & \leq & k \cdot e^{\max(w_2, \mathrm{dist}_{\mathbb{H}^2}(u_{k-1},u_k)-r)/2}, \end{array}$$

the last inequality holding provided we chose λ_0 sufficiently small. ■

Recall that $C_k^{\mathbf{II}}, C_k^{\mathbf{III}}, C_k^{\mathbf{IV},v}$ denote the number of black chunks starting from the origin, with length k and final pseudo-edge of type **II**, respectively type **III**, respectively type **IV** and final pseudo-edge of length $\in [r+v, r+v+1)$.

We now need to introduce analogous notations to deal with linked sequences of chunks. We set

$$S_{n,k}^{\mathbf{II}} := \left| \left\{ \begin{array}{l} \text{black linked sequences of chunks starting from } o\text{, consisting of precisely } n \text{ chunks,} \\ \text{having a last chunk of length } k \text{ and final pseudo-edge of type } \mathbf{II} \end{array} \right\} \right|.$$

We let $S_{n,k}^{\mathbf{III}}$ and $S_{n,k}^{\mathbf{IV},v}$ be defined analogously. Of course

$$S_{1,k}^{\mathbf{II}} = C_k^{\mathbf{II}}, \quad S_{1,k}^{\mathbf{III}} = C_k^{\mathbf{III}}, \quad S_{1,k}^{\mathbf{IV},v} = C_k^{\mathbf{IV},v}. \tag{27}$$

In particular, Corollary 50 provides bounds on the expectations $\mathbb{E}S_{n,k}^{\mathbf{II}}, \mathbb{E}S_{n,k}^{\mathbf{III}}$ and $S_{n,k}^{\mathbf{IV},v}$ when $n = 1$. The next lemma provides a system of recursive inequalities that will allow us to also bound these expectations for $n \geq 2$. From now on, it will be convenient to assume the parameter w_2 is an integer. Note that so far the only result that puts any restrictions on the value of w_2 is Corollary 47 which just states it has to be taken sufficiently large.

Lemma 56 *For every $w_1, \vartheta > 0$ and $w_2 \in \mathbb{N}$ there exists a $\lambda_0 = \lambda_0(w_1, w_2, \vartheta)$ such that for all $0 < \lambda < \lambda_0$ and $p \leq 10\lambda$, setting $r := 2\ln(1/\lambda)$, the following holds.*

For all $n \geq 1$, writing

$$\Sigma_n := 10^7 \cdot \sum_{\ell=1}^{\infty} \ell \cdot \left(e^{2w_2} \cdot \mathbb{E}S_{n,\ell}^{\mathbf{II}} + e^{2w_2} \cdot \mathbb{E}S_{n,\ell}^{\mathbf{III}} + \sum_{v=w_2}^{\infty} e^{2v+2} \cdot \mathbb{E}S_{n,\ell}^{\mathbf{IV},v} \right),$$

we have

$$\begin{array}{rcl} \mathbb{E}S_{n+1,1}^{\mathbf{II}} & \leq & \Sigma_n \cdot 10^3 \cdot e^{-w_1} \\ \mathbb{E}S_{n+1,2}^{\mathbf{III}} & \leq & \Sigma_n \cdot 10^3 \cdot \vartheta \cdot e^{2w_2} \\ \mathbb{E}S_{n+1,1}^{\mathbf{IV},v} & \leq & \Sigma_n \cdot 10^6 \cdot e^{v} \cdot e^{-e^{v/2}} \end{array}$$

and

$$\mathbb{E}S_{n+1,k}^{\tau} \leq \Sigma_n \cdot \mathbb{E}C_{k-1}^{\tau},$$

for τ one of $\mathbf{II}, \mathbf{III}$ or $(\mathbf{IV}, v)$ with $v \geq w_2$, and for all $k \geq 3$ when $\tau = \mathbf{III}$ and all $k \geq 2$ otherwise.

Proof. We'll need to introduce even more notation. We let

$$\mathcal{T} := \{\mathbf{II}, \mathbf{III}, (\mathbf{IV}, v) : v = w_2, w_2+1, \dots\}, \quad \mathcal{L} := \{a, b, c\},$$

denote the possible *types* of the final pseudo-edges of chunks, respectively types of links between consecutive chunks – corresponding to the cases described in part **(ii)** of Definition 30.

For $n \in \mathbb{N}$ and $\underline{k} \in \mathbb{N}^n, \underline{\tau} \in \mathcal{T}^n, \underline{t} \in \mathcal{L}^{n-1}$ we write $S_{\underline{k},\underline{\tau},\underline{t}}$ for the number of black, linked sequences of chunks $P_1, \dots, P_n$ starting from o, such that P_i has length k_i and final pseudo-edge of type τ_i (for all $i = 1, \dots, n$), and such that the link between P_i and P_{i+1} corresponds to case t_i of part **(ii)** of Definition 30 (for $i = 1, \dots, n = 1$). Clearly we can write for each $n, k \geq 1$ and $\tau \in \mathcal{T}$:

$$S^{\tau}_{n,k} = \sum_{\substack{\underline{k} \in \mathbb{N}^n, \underline{\tau} \in \mathcal{T}^n, \underline{t} \in \mathcal{L}^{n-1}, \\ \tau_n = \tau, k_n = k}} S_{\underline{k},\underline{\tau},\underline{t}}. \tag{28}$$

We will first derive the last inequality claimed in the lemma statement. Let us fix $n, k, \ell \in \mathbb{N}$ and $\sigma, \tau \in \mathcal{T}$ such that either $k \geq 2$ in case $\tau \neq \mathbf{III}$ or $k \geq 3$ in case $\tau = \mathbf{III}$. We pick vectors $\underline{k} \in \mathbb{N}^n, \underline{\tau} \in \mathcal{T}^n, \underline{t} \in \mathcal{L}^{n-1}$ satisfying $k_{n-1} = \ell, k_n = k$ and $\tau_{n-1} = \sigma, \tau_n = \tau$.

The parameters $\underline{k}, \underline{\tau}, \underline{t}$ determine the number of vertices in any of the linked sequences of pseudopaths counted by $S_{\underline{k},\underline{\tau},\underline{t}}$. For any such sequence of pseudopaths we have $V(P_1) \cup \dots \cup V(P_n) = \{o, z_1, \dots, z_m\}$ with $z_1, \dots, z_m \in \mathcal{Z}_{\text{b}}$ and $k_1 + \dots + k_n \leq m \leq k_1 + \dots + k_n + n - 1$. (To be precise $m = k_1 + \dots + k_n + |\{1 \leq i \leq n-1 : t_i \neq a\}|$ as $V(P_i) = k_i + 1$ for all i but every time $t_i = a$ the last vertex of P_i coincides with the first vertex of P_{i+1}.) By Corollary 5 we can write, for $m = m(\underline{k}, \underline{\tau}, \underline{t})$ as determined by the parameters

$$\mathbb{E}S_{\underline{k},\underline{\tau},\underline{t}} = (p\lambda)^m \int_{\mathbb{D}} \cdots \int_{\mathbb{D}} \mathbb{E}\left[g_{\underline{k},\underline{\tau},\underline{t}}(z_1, \dots, z_m; \mathcal{Z} \cup \{z_1, \dots, z_m\})\right] f(z_1) \dots f(z_m) \,\mathrm{d}\, z_1 \dots \mathrm{d}\, z_m,$$

where f is given by (2) and $g_{\underline{k},\underline{\tau},\underline{t}}$ is the indicator function that $o, z_1, \dots, z_m$ form a linked sequence of pseudopaths of the required kind wrt. the point set $\mathcal{Z} \cup \{o, z_1, \dots, z_m\}$.

The vertices of P_n are $z_{m-k}, \dots, z_m$. If $t_{n-1} = a$ then z_m is both the first vertex of P_n and also the last vertex of P_{n-1}. Recall that $\operatorname{cert}(z_{m-k+1}, \dots, z_m)$ needs to be disjoint from $\operatorname{cert}(P_1) \cup \dots \cup \operatorname{cert}(P_{n-1})$ (and this set is completely determined by $z_1, \dots, z_{m-k}$), otherwise $g_{\underline{k},\underline{\tau},\underline{t}}$ will equal zero. So, provided $t_{n-1} = a$ we can write, for every $z_1, \dots, z_m \in \mathbb{D}$:

$$\begin{gathered} \mathbb{E}\left[g_{\underline{k},\underline{\tau},\underline{t}}(z_1, \dots, z_m; \mathcal{Z} \cup \{z_1, \dots, z_m\})\right] \\ \leq \\ \mathbb{E}\left[g_{(k_1,\dots,k_{n-1});(\tau_1,\dots,\tau_{n-1});(t_1,\dots,t_{n-2})}(z_1, \dots, z_{m-k}; \mathcal{Z} \cup \{z_1, \dots, z_{m-k}\})\right] \cdot \\ \mathbb{E}\left[g_{C^{\tau}_{k-1}}(z_{m-k+1}, \dots, z_m; \mathcal{Z} \cup \{z_{m-k+1}, \dots, z_m\})\right] \cdot \\ 1_{\{\operatorname{dist}_{\mathbb{H}^2}(z_{m-k}, z_{m-k+1}) < r + w_2\}}, \end{gathered} \tag{29}$$

where $g_{C^{\tau}_{k-1}}$ is the indicator function that $z_{m-k+1}, \dots, z_m$ forms a chunk of length $k-1$ and with final edge of type τ (with respect to the point set $\mathcal{Z} \cup \{z_{m-k+1}, \dots, z_m\}$). Here we use that $g_{\underline{k},\underline{\tau},\underline{t}} = 0$ unless the position of $z_1, \dots, z_m$ is such that $\operatorname{cert}(z_{m-k+1}, \dots, z_m) \cap \operatorname{cert}(P_i) = \emptyset$ for $i = 1, \dots, n-1$; and if $z_1, \dots, z_m$ are such that all these intersections are empty then the event that $g_{(k_1,\dots,k_{n-1});(\tau_1,\dots,\tau_{n-1});(t_1,\dots,t_{n-2})}(z_1, \dots, z_{m-k}; \mathcal{Z} \cup \{z_1, \dots, z_{m-k}\}) = 1$ and the event that $g_{C^{\tau}_{k-1}}(z_{m-k+1}, \dots, z_m; \mathcal{Z} \cup \{z_{m-k+1}, \dots, z_m\}) = 1$ are independent.

We can write

$$\begin{aligned}
\mathbb{E}S_{\underline{k},\underline{\tau},\underline{t}} \quad \leq \quad & (p\lambda)^{m-k} \int_{\mathbb{D}} \cdots \int_{\mathbb{D}} \\
& \mathbb{E}\left[g_{(k_1,\ldots,k_{n-1});(\tau_1,\ldots,\tau_{n-1});(t_1,\ldots,t_{n-2})}(z_1,\ldots,z_{m-k};\mathcal{Z}\cup\{z_1,\ldots,z_{m-k}\})\right]\cdot \\
& \Bigg(p\lambda \int_{\mathbb{D}} 1_{\{\mathrm{dist}_{\mathbb{H}^2}(z_{m-k},z_{m-k+1})<r+w_2\}} \cdot \Bigg((p\lambda)^{k-1}\int_{\mathbb{D}} \cdots \int_{\mathbb{D}} \\
& \quad \mathbb{E}\left[g_{C^{\tau}_{k-1}}(z_{m-k+1},\ldots,z_m;\mathcal{Z}\cup\{z_{m-k+1},\ldots,z_m\})\right]\cdot \\
& \quad f(z_{m-k+1})\ldots f(z_m)\,\mathrm{d}\,z_m\ldots\mathrm{d}\,z_{m-k+1}\Bigg) f(z_{m-k+1})\,\mathrm{d}\,z_{m-k+1}\Bigg)\cdot \\
& \quad f(z_1)\ldots f(z_{m-k})\,\mathrm{d}\,z_{m-k}\ldots\mathrm{d}\,z_1
\end{aligned} \tag{30}$$

For every fixed $z_1,\ldots,z_{m-k+1}$, applying an isometry that maps z_{m-k+1} to o and Corollary 5, we have

$$\begin{gathered}
(p\lambda)^{k-1}\int_{\mathbb{D}} \cdots \int_{\mathbb{D}} \mathbb{E}\left[g_{C^{\tau}_{k-1}}(z_{m-k+1},\ldots,z_m;\mathcal{Z}\cup\{z_{m-k+1},\ldots,z_m\})\right]\cdot \\
f(z_{m-k+2})\ldots f(z_m)\,\mathrm{d}\,z_m\ldots\mathrm{d}\,z_{m-k+2} \\
= \\
(p\lambda)^{k-1}\int_{\mathbb{D}} \cdots \int_{\mathbb{D}} \mathbb{E}\left[g_{C^{\tau}_{k-1}}(o,z_1,\ldots,z_{k-1};\mathcal{Z}\cup\{o,z_1,\ldots,z_{k-1}\})\right]\cdot \\
f(z_1)\ldots f(z_{k-1})\,\mathrm{d}\,z_1\ldots\mathrm{d}\,z_{k-1} \\
= \\
\mathbb{E}C^{\tau}_{k-1}.
\end{gathered} \tag{31}$$

Of course, for every fixed $z_{m-k}\in\mathbb{D}$, we have

$$\begin{aligned}
p\lambda\int_{\mathbb{D}} 1_{\{\mathrm{dist}_{\mathbb{H}^2}(z_{m-k},z_{m-k+1})<r+w_2\}} f(z_{m-k+1})\,\mathrm{d}\,z_{m-k+1} \quad &= \quad p\lambda\cdot\mathrm{area}_{\mathbb{H}^2}(B_{\mathbb{H}^2}(o,r+w_2)) \\
&\leq \quad 10^3\cdot e^{w_2},
\end{aligned}$$

by Lemma 20. It follows that:

$$\begin{aligned}
\mathbb{E}S_{\underline{k},\underline{\tau},\underline{t}} \quad \leq \quad & 10^3\cdot e^{w_2}\cdot\mathbb{E}C^{\tau}_{k-1}\cdot(p\lambda)^{m-k}\int_{\mathbb{D}} \cdots \int_{\mathbb{D}} \\
& \mathbb{E}\left[g_{(k_1,\ldots,k_{n-1});(\tau_1,\ldots,\tau_{n-1});(t_1,\ldots,t_{n-2})}(z_1,\ldots,z_{m-k};\mathcal{Z}\cup\{z_1,\ldots,z_{m-k}\})\right]\cdot \\
& f(z_1)\ldots f(z_{m-k})\,\mathrm{d}\,z_{m-k}\ldots\mathrm{d}\,z_1 \\
= \quad & 10^3\cdot e^{w_2}\cdot\mathbb{E}C^{\tau}_{k-1}\cdot\mathbb{E}S_{(k_1,\ldots,k_n),(\tau_1,\ldots,\tau_{n-1}),(t_1,\ldots,t_{n-2})}.
\end{aligned} \tag{32}$$

(Provided $t_{n-1}=a$.)

If $t_{n-1}=b$ then z_{m-k}, the first vertex of P_n, does not lie on P_{n-1} and we need that $\mathrm{dist}_{\mathbb{H}^2}(z_{m-k},z_{m-k+1})<r+w_2$ and $\mathrm{cert}(z_{m-k},z_{m-k+1})$ intersects $\mathrm{cert}(P_{n-1})=\mathrm{cert}(z_{m-k_{n-1}-k-2},\ldots,z_{m-k-1})$. So instead of (29) we can now write

$$\begin{gathered}\mathbb{E}\left[g_{\underline{k},\underline{\tau},\underline{t}}(z_1,\ldots,z_m;\mathcal{Z}\cup\{z_1,\ldots,z_m\})\right]\\ \leq\\ \mathbb{E}\left[g_{(k_1,\ldots,k_{n-1});(\tau_1,\ldots,\tau_{n-1});(t_1,\ldots,t_{n-1})}(z_1,\ldots,z_{m-k};\mathcal{Z}\cup\{z_1,\ldots,z_{m-k}\})\right]\cdot\\ \mathbb{E}\left[g_{C^{\tau}_{k-1}}(z_{m-k+1},\ldots,z_m;\mathcal{Z}\cup\{z_{m-k+1},\ldots,z_m\})\right]\cdot\\ 1_{\left\{\substack{\mathrm{cert}(z_{m-k},z_{m-k+1})\cap\mathrm{cert}(z_{m-k_{n-1}-k-2},\ldots,z_{m-k-1})\neq\emptyset,\\ \mathrm{dist}_{\mathbb{H}^2}(z_{m-k},z_{m-k+1})<r+w_2}\right\}}\end{gathered}\tag{33}$$

Arguing as in (30) and reusing (31) we find that

$$\begin{aligned}\mathbb{E}S_{\underline{k},\underline{\tau},\underline{t}} \leq\ & \mathbb{E}C^{\tau}_{k-1}\cdot(p\lambda)^{m-(k+1)}\int_{\mathbb{D}}\cdots\int_{\mathbb{D}}\\ & \mathbb{E}\left[g_{(k_1,\ldots,k_{n-1});(\tau_1,\ldots,\tau_{n-1});(t_1,\ldots,t_{n-2})}(z_1,\ldots,z_{m-k};\mathcal{Z}\cup\{z_1,\ldots,z_{m-k-1}\})\right]\cdot\\ & \Bigg((p\lambda)^2\int_{\mathbb{D}}\int_{\mathbb{D}}1_{\left\{\substack{\mathrm{cert}(z_{m_k},z_{m-k+1})\cap\mathrm{cert}(z_{m-k_{n-1}-k-2},\ldots,z_{m-k-1})\neq\emptyset,\\ \mathrm{dist}_{\mathbb{H}^2}(z_{m_k},z_{m-k+1})<r+w_2}\right\}}\cdot\\ & f(z_{m-k})f(z_{m-k+1})\,\mathrm{d}\,z_{m-k}\,\mathrm{d}\,z_{m-k+1}\Bigg)\cdot\\ & f(z_1)\ldots f(z_{m-k})\,\mathrm{d}\,z_{m-k}\ldots\mathrm{d}\,z_1\end{aligned}$$

For any fixed $z_1,\ldots,z_{m-k-1}\in\mathbb{D}$ we have that if $z_{m-k_{n-1}-k-2},\ldots,z_{m-k-1}$ is not a pre-chunk then $g_{(k_1,\ldots,k_{n-1});(\tau_1,\ldots,\tau_{n-1});(t_1,\ldots,t_{n-2})}(z_1,\ldots,z_{m-k};\mathcal{Z}\cup\{z_1,\ldots,z_{m-k-1}\})=0$ and otherwise we can apply Lemma 51 to get

$$\begin{gathered}(p\lambda)^2\int_{\mathbb{D}}\int_{\mathbb{D}}1_{\left\{\substack{\mathrm{cert}(z_{m_k},z_{m-k+1})\cap\mathrm{cert}(z_{m-k_{n-1}-k-2},\ldots,z_{m-k-1})\neq\emptyset,\\ \mathrm{dist}_{\mathbb{H}^2}(z_{m_k},z_{m-k+1})<r+w_2}\right\}}\cdot\\ f(z_{m-k})f(z_{m-k+1})\,\mathrm{d}\,z_{m-k}\,\mathrm{d}\,z_{m-k+1}\\ \leq\\ 10^5\cdot k_{n-1}\cdot e^{2v(\tau_{n-1})+2}\\ \leq\\ 10^6\cdot k_{n-1}\cdot e^{2v(\tau_{n-1})},\end{gathered}$$

where $v(\sigma)=w_2$ if $\sigma\in\{\mathbf{II},\mathbf{III}\}$ and otherwise $v(\sigma)$ is determined via $\sigma=:(\mathbf{IV},v(\sigma)-1)$. (That is, if $\sigma=(\mathbf{IV},x)$ for some $x\geq w_2$ then $v(\sigma)=x+1$.) We conclude that if $t_{n-1}=b$ then

$$\mathbb{E}S_{\underline{k},\underline{\tau},\underline{t}}\leq 10^6\cdot k_{n-1}\cdot e^{2v(\tau_{n-1})}\cdot\mathbb{E}C^{\tau}_{k-1}\cdot\mathbb{E}S_{(k_1,\ldots,k_{n-1});(\tau_1,\ldots,\tau_{n-1});(t_1,\ldots,t_{n-2})}.\tag{34}$$

If $t_{n-1}=c$ then we replace (33) with

$$
\begin{gathered}
\mathbb{E}\left[g_{\underline{k},\underline{\tau},\underline{t}}(z_1,\ldots,z_m;\mathcal{Z}\cup\{z_1,\ldots,z_m\})\right]\\
\le\\
\mathbb{E}\left[g_{(k_1,\ldots,k_{n-1});(\tau_1,\ldots,\tau_{n-1});(t_1,\ldots,t_{n-1})}(z_1,\ldots,z_{m-k};\mathcal{Z}\cup\{z_1,\ldots,z_{m-k}\})\right]\cdot\\
\mathbb{E}\left[g_{C^{\tau}_{k-1}}(z_{m-k+1},\ldots,z_m;\mathcal{Z}\cup\{z_{m-k+1},\ldots,z_m\})\right]\cdot\\
1\left\{\begin{array}{l}\mathrm{dist}_{\mathbb{H}^2}(z_{m-k},\mathrm{cert}(z_{m-k_{n-1}-k-2},\ldots,z_{m-k-1}))<r/1000,\\ \mathrm{dist}_{\mathbb{H}^2}(z_{m-k},z_{m-k+1})<r+w_2,\\ \mathrm{cert}(z_{m-k},z_{m-k+1})\cap\mathrm{cert}(z_{m-k_{n-1}-k-2},\ldots,z_{m-k-1})=\emptyset,\\ z_{m-k}z_{m-k+1}\text{ is a pseudoedge.}\end{array}\right\}\cdot
\end{gathered}
\tag{35}
$$

Arguing as before, but using Lemma 52 in place of Lemma 51 gives:

$$
\mathbb{E}S_{\underline{k},\underline{\tau},\underline{t}}\le k_{n-1}\cdot e^{v(\tau_{n-1})/2}\cdot\mathbb{E}C^{\tau}_{k-1}\cdot\mathbb{E}S_{(k_1,\ldots,k_{n-1});(\tau_1,\ldots,\tau_{n-1});(t_1,\ldots,t_{n-2})}.\tag{36}
$$

(Provided $t_{n-1}=c$.)

Combining (28) and (32), (34), (36) gives

$$
\begin{aligned}
\mathbb{E}S^{\tau}_{n,k} &= \sum_{\substack{\underline{k}\in\mathbb{N}^n,\underline{\tau}\in\mathfrak{T}^n,\underline{t}\in\mathcal{L}^{n-1},\\ \tau_n=\tau,k_n=k}}\mathbb{E}S_{\underline{k},\underline{\tau},\underline{t}}\\
&\le \sum_{\substack{k_1,\ldots,k_{n-1}\in\mathbb{N}\\ \tau_1,\ldots,\tau_{n-1}\in\mathfrak{T}\\ t_1,\ldots,t_{n-2}\in\mathcal{L}}}\mathbb{E}C^{\tau}_{k-1}\cdot\left(10^3e^{w_2}+10^6\cdot k_{n-1}\cdot e^{2v(\tau_{n-1})}+k_{n-1}\cdot e^{v(\tau_{n-1})/2}\right)\cdot\\
&\qquad\qquad\mathbb{E}S_{(k_1,\ldots,k_{n-1});(\tau_1,\ldots,\tau_{n-1});(t_1,\ldots,t_{n-2})}\\
&\le \sum_{\substack{k_1,\ldots,k_{n-1}\in\mathbb{N}\\ \tau_1,\ldots,\tau_{n-1}\in\mathfrak{T}\\ t_1,\ldots,t_{n-2}\in\mathcal{L}}}\mathbb{E}C^{\tau}_{k-1}\cdot 10^7\cdot k_{n-1}\cdot e^{2v(\tau_{n-1})}\cdot\mathbb{E}S_{(k_1,\ldots,k_{n-1});(\tau_1,\ldots,\tau_{n-1});(t_1,\ldots,t_{n-2})}\\
&= 10^7\cdot\mathbb{E}C^{\tau}_{k-1}\cdot\sum_{\ell=1}^{\infty}\ell\cdot\sum_{\sigma\in\mathfrak{T}}e^{2v(\sigma)}\cdot\sum_{\substack{k_1,\ldots,k_{n-1}\in\mathbb{N}\\ \tau_1,\ldots,\tau_{n-1}\in\mathfrak{T}\\ t_1,\ldots,t_{n-2}\in\mathcal{L}\\ k_{n-1}=\ell,\tau_{n-1}=\sigma}}\mathbb{E}S_{(k_1,\ldots,k_{n-1});(\tau_1,\ldots,\tau_{n-1});(t_1,\ldots,t_{n-2})}\\
&= \mathbb{E}C^{\tau}_{k-1}\cdot 10^7\cdot\sum_{\ell=1}^{\infty}\ell\cdot\sum_{\sigma\in\mathfrak{T}}e^{2v(\sigma)}\cdot\mathbb{E}S^{\sigma}_{n-1,\ell}\\
&= \mathbb{E}C^{\tau}_{k-1}\cdot 10^7\cdot\sum_{\ell=1}^{\infty}\ell\cdot\left(e^{2w_2}\cdot\mathbb{E}S^{\mathbf{II}}_{n-1,\ell}+e^{2w_2}\cdot\mathbb{E}S^{\mathbf{III}}_{n-1,\ell}+\sum_{v\ge w_2}e^{2v+2}\cdot\mathbb{E}S^{\mathbf{IV},v}_{n-1,\ell}\right)\\
&= \mathbb{E}C^{\tau}_{k-1}\cdot\Sigma_n,
\end{aligned}
$$

establishing the last inequality in the lemma statement.

The first inequality (the case when $k=1$ and $\tau=\mathbf{II}$) follows analogously, replacing $g_{C^{\tau}_{k-1}}$ in (29), (33), (35) by $1_{\{\mathrm{dist}_{\mathbb{H}^2}(z_{m-1},z_m)<r-w_1\}}$ and applying Lemma 20 to the innermost integral in the analogues of (30).

The second inequality (the case when $k = 2$ and $\tau = \mathbf{III}$) follows in the same way, now using the indicator function that $\mathrm{dist}_{\mathbb{H}^2}(z_{m-2}, z_{m-1}), \mathrm{dist}_{\mathbb{H}^2}(z_{m-1}, z_m) < r + w_2$ and $\angle z_{m-2} z_{m-1} z_m < \vartheta$ and applying Lemma 21.

For the case when $k = 1$ and $\tau = (\mathbf{IV}, v)$ we replace $g_{C^{\tau}_{k-1}}$ by the indicator function corresponding to $F_{\text{IV-a},\geq v}$ from Lemma 53 in case $t_{n-1} = a$; by the indicator function corresponding to $F_{\text{IV-b},\geq v}$ from Lemma 54 in case $t_{n-1} = b$; and the indicator function corresponding to $F_{\text{IV-c},\geq v}$ form Lemma 55 in case $t_{n-1} = c$; and then apply Lemma 53, respectively Lemma 54, respectively Lemma 55 to the innermost integral in the analogues of (30). ■

Lemma 57 *For every $0 < \varepsilon < 1$ and $c > 0$ there exist $\lambda_0, w_1, \vartheta > 0$ and $w_2 > c$ such that $w_2 \in \mathbb{N}$ and for all $0 < \lambda < \lambda_0$ and $p \leq (1 - \varepsilon) \cdot (\pi/3) \cdot \lambda$, setting $r := 2\ln(1/\lambda)$, almost surely there are no infinite, black, linked sequences of chunks starting from the origin o.*

Proof. We will make use of the bounds provided by Lemma 46, Corollary 50 and Lemma 56. We will choose the integer $w_2 > c$ so large that $10^4 e^{w_2} e^{-w_2/2} < \varepsilon/6$. So in particular, for λ sufficiently small, the expected number of good, black pseudopaths starting from the origin satisfies

$$\mathbb{E}G_k \leq (1 - \varepsilon/3)^k.$$

Let Σ_n be as in the statement of Lemma 56. By the observation (27) and Corollary 50, we have

$$\begin{aligned}
\Sigma_1 &= 10^7 \cdot \sum_{\ell=1}^{\infty} \ell \cdot \left(e^{2w_2} \cdot \mathbb{E}C_\ell^{\mathbf{II}} + e^{2w_2} \cdot \mathbb{E}C_\ell^{\mathbf{III}} + \sum_{v \geq w_2} e^{2v} \cdot \mathbb{E}C_\ell^{\mathbf{IV},v} \right) \\
&\leq 10^7 \cdot \left(10^3 \cdot e^{2w_2} \cdot \left(e^{-w_1} + \vartheta e^{w_2} \right) + 10^4 \cdot \sum_{v=w_2}^{\infty} e^{(5/2)v} e^{-e^{v/2}} \right) \cdot \left(\sum_{\ell=1}^{\infty} \ell \cdot (1 - \varepsilon/3)^{\ell-1} \right) \\
&< \infty.
\end{aligned}$$

The recursive relations given by Lemma 56 show that for all $n \geq 1$:

$$
\begin{aligned}
\Sigma_{n+1} &\leq \Sigma_n \cdot 10^7 \cdot \Bigg(e^{2w_2} \cdot 10^3 \cdot e^{-w_1} + 2e^{2w_2} \cdot 10^3 \cdot \vartheta \cdot e^{w_2} + 10^6 \cdot \sum_{v=w_2} e^{3v} e^{-e^{v/2}} + \\
&\qquad \sum_{\ell=2}^{\infty} e^{2w_2} \cdot \ell \cdot \mathbb{E} C^{\mathbf{II}}_{\ell-1} + \sum_{\ell=3}^{\infty} e^{2w_2} \cdot \ell \cdot \mathbb{E} C^{\mathbf{III}}_{\ell-1} + \sum_{\ell=2}^{\infty} e^{2v} \cdot \ell \cdot \mathbb{E} C^{\mathbf{IV},v}_{\ell-1} \Bigg) \\
&\leq \Sigma_n \cdot 10^7 \cdot \Bigg(10^3 \cdot e^{2w_2 - w_1} + 2 \cdot 10^3 \cdot \vartheta \cdot e^{3w_2} + 10^6 \cdot \sum_{v=w_2}^{\infty} e^{3v} e^{-e^{v/2}} + \\
&\qquad \left(10^3 \cdot e^{2w_2 - w_1} + 10^3 \cdot \vartheta \cdot e^{3w_2} + 10^4 \cdot \sum_{v=w_2}^{\infty} e^{(5/2)v} \cdot e^{-e^{v/2}} \right) \cdot \left(\sum_{\ell=2}^{\infty} \ell \cdot (1 - \varepsilon/3)^{\ell-2} \right) \Bigg) \\
&\leq \Sigma_n \cdot 10^{13} \cdot \left(e^{2w_2 - w_1} + \vartheta \cdot e^{3w_2} + \sum_{v=w_2}^{\infty} e^{3v} e^{-e^{v/2}} \right) \cdot \left(1 + \sum_{\ell=2}^{\infty} \ell \cdot (1 - \varepsilon/3)^{\ell-2} \right) \\
&= \Sigma_n \cdot 10^{13} \cdot \left(e^{2w_2 - w_1} + \vartheta \cdot e^{3w_2} + \sum_{v=w_2}^{\infty} e^{3v} e^{-e^{v/2}} \right) \cdot \left(\frac{9 + 3\varepsilon + \varepsilon^2}{\varepsilon^2} \right) \\
&\leq \Sigma_n \cdot 10^{15} \cdot \varepsilon^{-2} \cdot \left(e^{2w_2 - w_1} + \vartheta \cdot e^{3w_2} + \sum_{v=w_2}^{\infty} e^{3v} e^{-e^{v/2}} \right)
\end{aligned}
$$

For the sake of the presentation, we introduce an additional small constant $\delta > 0$. Without loss of generality we can assume chose $w_2 > c$ large enough so that

$$
\sum_{v=w_2}^{\infty} e^{3v} \cdot e^{-e^{v/2}} < \delta.
$$

We can also assume w_1, ϑ are such that $e^{2w_2 - w_1}, \vartheta e^{3w_2} < \delta$. Filling these bounds into the bound on Σ_{n+1} gives

$$
\Sigma_{n+1} \leq \Sigma_n \cdot 10^{15} \cdot \left(\frac{3\delta}{\varepsilon^2} \right) \leq \Sigma_n \cdot \left(\frac{1}{2} \right),
$$

the last line holding because we have chosen δ appropriately.

To recap, we can choose w_1, w_2, ϑ in such a way that $\Sigma_n \leq \left(\frac{1}{2}\right)^{n-1} \cdot \Sigma_1$ for all n; and $\Sigma_1 < \infty$. In particular $\Sigma_n \to 0$ as $n \to \infty$.

Let us denote by S_n the total number of black, linked sequences of chunks starting from the origin and consisting of precisely n chunks. Obviously

$$
\mathbb{E} S_n = \sum_{k=1}^{\infty} \mathbb{E} S^{\mathbf{II}}_{n,k} + \sum_{k=1}^{\infty} \mathbb{E} S^{\mathbf{III}}_{n,k} + \sum_{k=1}^{\infty} \sum_{v=w_2}^{\infty} \mathbb{E} S^{\mathbf{IV},v}_{n,k} \leq \Sigma_n.
$$

But then we also have that

$$
\mathbb{E} S_n \xrightarrow[n \to \infty]{} 0.
$$

It follows that, almost surely, there are is no infinite, black, linked sequence of chunks starting from o. ■

Combining Proposition 31, Corollary 47 and Lemma 57 immediately gives:

Corollary 58 *For every $\varepsilon > 0$ there is a $\lambda_0 = \lambda_0(\varepsilon)$ such that for all $0 < \lambda < \lambda_0$ and all $p \leq (1-\varepsilon) \cdot (\pi/3) \cdot \lambda$, almost surely, the black cluster of o in the Voronoi tessellation for $\mathcal{Z} \cup \{o\}$ is finite.*

For completeness we point out how Proposition 28 follows from Corollary 58.

Proof of Proposition 28. This follows from Corollary 58 by a near verbatim repeat of the proof of Corollary 47. We only mention the changes that need to be made. Now, we let N_x be the number of $z \in \mathcal{Z}_b \cap B_{\mathbb{H}^2}(o,x)$ that are part of an infinite black component of the Voronoi tessellation, and we let g be the indicator function that $z \in B_{\mathbb{H}^2}(o,x)$ and that z lies in an infinite, black cluster in the Voronoi tessellation for $\mathcal{Z} \cup \{z\}$. ■

4 Suggestions for further work

Our Theorem 1 states that $p_c(\lambda) = (\pi/3)\lambda + o(\lambda)$ as $\lambda \searrow 0$, answering a question of Benjamini and Schramm [9]. A natural direction for research is to try and find more terms in the expansion.

Problem 59 *Determine a constant c such that $p_c(\lambda) = (\pi/3)\lambda + c\lambda^2 + o(\lambda^2)$ as $\lambda \searrow 0$, or show no such constant exists.*

In our proofs we have either taken $p \geq (1-\varepsilon) \cdot (\pi/3) \cdot \lambda$, for the upper bound on $p_c(\lambda)$, or $p \leq (1-\varepsilon) \cdot (\pi/3) \cdot \lambda$, for the lower bound. We have always taken ε constant in our paper and have not made any effort to see how fast we can send ε to zero as a function of λ before our proof technique breaks down. Of course many of our bounds are rather crude and it seems likely that more fine-grained proof techniques will need to be developed.

As mentioned in the introduction, the critical value for percolation in $\mathbb{Z}^d$ for high dimension d is related, at least on an intuitive level, to our result. And, as also mentioned, a trivial comparison to branching processes shows the critical value for bond percolation on $\mathbb{Z}^d$ satisfies $p_c(\mathbb{Z}^d) \geq \frac{1}{2d-1} > \frac{1}{2d}$, the reciprocal of the degree. (It was in fact shown by Van der Hofstad and Slade [48] that $p_c(\mathbb{Z}^d) = \frac{1}{2d} + \frac{1}{4}d^{-2} + \frac{7}{16}d^{-3} + O(d^{-4})$. So we even have $p_c(\mathbb{Z}^d) > \frac{1}{2d-1}$ for large enough d.) Inspired by this, and the fact that the typical degree is actually strictly larger that $\frac{3}{\pi\lambda}$, we offer the following conjecture:

Conjecture 60 *There exists a $\lambda_0 > 0$ such that $p_c(\lambda) > (\pi/3) \cdot \lambda$ for all $0 < \lambda < \lambda_0$.*

Of course the answer to Problem 59 is likely to also tell us whether this conjecture holds or not. But, perhaps Conjecture 60 can be settled via a different route.

We reiterate a natural conjecture of Benjamini and Schramm.

Conjecture 61 ([9]) *$p_c(\lambda)$ is strictly increasing.*

Another natural conjecture in the same vein is:

Conjecture 62 *$p_c(\lambda)$ is differentiable.*

We were tempted to write "smooth" in place of "differentiable", but opted against it to give whoever attempts to prove the conjecture the best chances.

Of course Poisson-Voronoi percolation can also be defined on d-dimensional hyperbolic space $\mathbb{H}^d$, and we expect that the main result of the current paper will generalize. In two dimensions, the typical degree is the same as the number of 1-faces of the typical cell. In d-dimensions the relevant corresponding quantity is the number of $(d-1)$-faces of the typical cell.

Conjecture 63 *For every d, as the intensity $\lambda \searrow 0$, the critical value for Poisson-Voronoi percolation on $\mathbb{H}^d$ is asymptotically equal to the reciprocal of the expected number of $(d-1)$-faces of the typical cell.*

It might be possible to leverage some of the existing work on the expected f-vectors of the typical cell in [13, 19, 26]. But, of course it might also be possible to prove or disprove the conjecture without knowing the expected number of $(d-1)$-faces precisely.

In a recent separate paper, we proved the conjecture of Benjamini and Schramm that $p_c(\lambda) \to 1/2$ as $\lambda \to \infty$ for planar, hyperbolic Poisson-Voronoi percolation. It seems natural to expect that this result generalizes to arbitrary dimensions.

Conjecture 64 *For every d, as the intensity $\lambda \to \infty$, the critical value for Poisson-Voronoi percolation on $\mathbb{H}^d$ tends to the critical value for Poisson-Voronoi percolation on $\mathbb{R}^d$.*

A complicating issue here is that for Poisson-Voronoi percolation on $\mathbb{R}^d$ there is a large gap between the best known lower and upper bounds [3, 4]. But, again, it may be possible to prove the conjecture without first determining the precise critical value in the Euclidean case.

Acknowledgements

We thank the anonymous referee for many helpful comments that have greatly improved the paper.

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