Logical limit laws for minor-closed classes of graphs

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Abstract

Let $\mathcal{G}$ be an addable, minor-closed class of graphs. We prove that the zero-one law holds in monadic second-order logic (MSO) for the random graph drawn uniformly at random from all *connected* graphs in $\mathcal{G}$ on n vertices, and the convergence law in MSO holds if we draw uniformly at random from all graphs in $\mathcal{G}$ on n vertices. We also prove analogues of these results for the class of graphs embeddable on a fixed surface, provided we restrict attention to first order logic (FO). Moreover, the limiting probability that a given FO sentence is satisfied is independent of the surface S. We also prove that the closure of the set of limiting probabilities is always the finite union of at least two disjoint intervals, and that it is the same for FO and MSO. For the classes of forests and planar graphs we are able to determine the closure of the set of limiting probabilities precisely. For planar graphs it consists of exactly 108 intervals, each of length $\approx 5 \cdot 10^{-6}$. Finally, we analyse examples of non-addable classes where the behaviour is quite different. For instance, the zero-one law does not hold for the random caterpillar on n vertices, even in FO.

1 Introduction

We say that a sequence $(G_n)_n$ of random graphs obeys the *zero-one law* with respect to some logical language L if for every sentence φ in L the probability that a graph G_n satisfies φ tends either to 0 or 1, as n goes to infinity. We say that $(G_n)_n$ obeys the *convergence law* with respect to L, if for every φ in L, the probability that G_n satisfies φ tends to a limit (not necessarily zero or one) as n tends to infinity.

The prime example of a logical language is the *first order language of graphs* (FO). Formulas in this language are constructed using variables $x, y, \dots$ ranging over the vertices of a graph, the usual quantifiers $\forall, \exists$, the usual logical connectives $\neg, \vee, \wedge$, etc., parentheses and the binary relations $=, \sim$, where $x \sim y$ denotes that x and y are adjacent. In FO one can for instance write "G is triangle-free" as $\neg\exists x, y, z : (x \sim y) \wedge (x \sim z) \wedge (y \sim z)$.

The classical example of a zero-one law is a result due to Glebskii et al. [19] and independently to Fagin [14], stating that the zero-one law holds when G_n is chosen uniformly at random among all $2^{\binom{n}{2}}$ labelled graphs on n vertices and the language is FO. The (non-)existence of FO-zero-one and convergence laws has been investigated more generally in the $G(n, p)$ binomial model, where there are n labelled vertices and edges are drawn independently with probability p (the case $p = 1/2$ of course corresponds to the uniform distribution on all labelled graphs on n vertices). Here, the FO-zero-one law holds for all constant p and in many other cases. In particular, a remarkable

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result of Shelah and Spencer says that if $p = n^{-\alpha}$ with $0 \leq \alpha \leq 1$ fixed, then the FO-zero-one law holds if and only if α is an irrational number [37]. What is more, when α is rational then the *convergence law* in fact fails. That is, for rational α Shelah and Spencer were able to construct an ingenious FO-sentence such that the probability that $G(n, n^{-\alpha})$ satisfies it oscillates between zero and one. The book [38] by Spencer contains a detailed account of the story of logical limit laws for the $G(n, p)$ binomial model.

There are also several results available on zero-one laws for random graphs that satisfy some global condition, such as being regular, having bounded degree or being H-free. One of the earliest instances of such results deals with K_{t+1}-free graphs: Kolaitis, Prömel and Rothschild [25] proved an FO-zero-one law for the graph chosen uniformly at random from all K_{t+1}-free graphs on n vertices. For random d-regular graphs, with d fixed, there cannot be a zero-one law. For instance, the probability of containing a triangle is expressible in FO and tends to a constant different from 0 and 1. Using the configuration model, Lynch [27] proved the FO-*convergence law* for graphs with a given degree sequence (subject to some conditions on this degree sequence), which in particular covers d-regular graphs with d fixed. For dense d-regular graphs, that is when $d = \Theta(n)$, a zero-one law was proved by Haber and Krivelevich [20], who were also able to obtain an analogue of the striking result of Shelah-Spencer mentioned above for random regular graphs.

In a different direction, McColm [28] considers trees sampled uniformly at random from all (labelled) trees on n vertices. He shows that a zero-one law holds in *monadic second order* (MSO) language of graphs, which is FO enriched with quantification over sets of vertices. That is, we now have additional variables $X, Y, \ldots$ ranging over sets of vertices and an additional binary relation $\in$, that allows us to ask whether $x \in X$, for x a vertex-variable and X a set-variable. This results in a stronger language which is able to express properties such as connectedness or k-colorability for any fixed k. In MSO we can for instance express "G is connected" as
$\forall X : (\forall x : x \in X) \vee \neg(\exists x : x \in X) \vee (\exists x, y : (x \in X) \wedge \neg(y \in X) \wedge (x \sim y))$.

The proof in [28] is based on two facts: for each $k > 0$ there exists a rooted tree T_k such that, if two trees A and B both have T_k as a rooted subtree, then A and B agree on all sentences of quantifier depth at most k (defined in Section 2.1 below); and the fact that with high probability a random tree contains T_k as a rooted subtree. We show that this approach can be adapted to a much more general setting, as we explain next.

A class $\mathcal{G}$ of graphs is *minor-closed* if every minor of a graph in $\mathcal{G}$ is also in $\mathcal{G}$. Every minor-closed class is characterized by the set of its excluded minors, which is finite by the celebrated Robertson-Seymour theorem. Notable examples of minor-closed graphs are forests, planar graphs and graphs embeddable on a fixed surface. We say that $\mathcal{G}$ is addable if it holds that **1)** $G \in \mathcal{G}$ if and only if all its components are in $\mathcal{G}$, and **2)** if G' is obtained from $G \in \mathcal{G}$ by adding an arbitrary edge between two separate components of G, then $G' \in \mathcal{G}$ also. Planar graphs constitute an addable class of graphs, but graphs embeddable on a surface other than the sphere may not. (A 5-clique is for instance embeddable on the torus, but the vertex-disjoint union of two 5-cliques is not – see [32, Theorem 4.4.2]). Other examples of addable classes are outerplanar graphs, series-parallel graphs, graphs with bounded tree-width, and graphs with given 3-connected components [18]. Relying heavily on results of McDiarmid [30], we are able to prove the MSO-zero-one law for the random graph chosen uniformly at random from all *connected* graphs on n vertices in an addable minor-closed class. If $\mathcal{G}$ is a class of graphs, then $\mathcal{C} \subseteq \mathcal{G}$ will denote the set of *connected* graphs from $\mathcal{G}$; the notation $\mathcal{G}_n$ will denote all graphs in $\mathcal{G}$ with vertex set $[n] := \{1, \ldots, n\}$, and $\mathcal{C}_n$ is defined analogously. If A is a finite set then $X \in_u A$ denotes that X is chosen uniformly at random from A.

Theorem 1.1 *Let $\mathcal{G}$ be an addable, minor-closed class of graphs and let $C_n \in_u \mathcal{C}_n$ be the random* connected *graph from $\mathcal{G}$. Then C_n obeys the* MSO*-zero-one law.*

It is worth mentioning that the MSO-zero-one law does not hold in $G(n, 1/2)$ – see [24]. For the "general" random graph from an addable class $\mathcal{G}$, i.e. $G_n \in_u \mathcal{G}_n$, there cannot be a zero-one law, even in FO logic. The reason is that there are sentences expressible in FO, such as the existence of an isolated vertex, that have a limiting probability strictly between 0 and 1. We are however able to prove the *convergence* law in this case.

Theorem 1.2 *Let $\mathcal{G}$ be an addable, minor-closed class of graphs and let $G_n \in_u \mathcal{G}_n$. Then G_n obeys the* MSO*-convergence law.*

The proof is based on the fact that with high probability there is a "giant" component of size $n - O(1)$, and uses the extraordinarily precise description of the limiting distribution of the *fragment*, the part of the graph that remains after we remove the largest component [30].

Moving away from addable classes of graphs, let S be a fixed surface and consider the class of graphs that can be embedded in S. In this case we prove a zero-one law in FO for connected graphs. The proof is based on recent results on random graphs embeddable on a surface [29, 9] and an application of Gaifman's locality theorem (see Theorem 2.8 below). The notation $G \models \varphi$ means "G satisfies φ".

Theorem 1.3 *Fix a surface S, let $\mathcal{G}$ be the class of all graphs embeddable on S and let $C_n \in_u \mathcal{C}_n$ be the random* connected *graph from $\mathcal{G}$. Then C_n obeys the* FO*-zero-one law. Moreover, the values of the limiting probabilities $\lim_{n\to\infty} \mathbb{P}(C_n \models \varphi)$ do not depend on the surface S.*

We remark that an analogous result was proved for random *maps* (connected graphs with a given embedding) on a fixed surface [3]. Again for arbitrary graphs we prove a convergence law in FO. Moreover, we show that the limiting probability of an FO sentence does not depend on the surface and is the same as for planar graphs.

Theorem 1.4 *Fix a surface S, let $\mathcal{G}$ be the class of all graphs embeddable on S and let $G_n \in_u \mathcal{G}_n$. Then G_n obeys the* FO*-convergence law. Moreover, the values of the limiting probabilities $\lim_{n\to\infty} \mathbb{P}(G_n \models \varphi)$ do not depend on the surface S.*

We conjecture that both Theorems 1.3 and 1.4 extend to MSO logic. See the last section of this paper for a more detailed discussion.

Having proved Theorem 1.2, a natural question is which numbers $p \in [0, 1]$ are limiting probabilities of some MSO sentence. The proof of Theorem 1.2 provides an expression for the limiting probabilities in terms of the so-called Boltzmann-Poisson distribution (defined in Section 2.2), but at present we are not able to deduce a complete description of the set of limiting probabilities from it. We are however able to derive some information on the structure of this set. Two easy observations are that, since there are only countably many sentences $\varphi \in \text{MSO}$, the set $\{\lim_{n\to\infty} \mathbb{P}(G_n \models \varphi) : \varphi \in \text{MSO}\}$ must obviously also be countable; and since $\lim_{n\to\infty} \mathbb{P}(G_n \models \neg\varphi) = 1 - \lim_{n\to\infty} \mathbb{P}(G_n \models \varphi)$, it is symmetric with respect to $1/2$.

Certainly not every number in $p \in [0, 1]$ is a limiting probability of an MSO sentence, as there are only countably many such limiting probabilities. A natural question is whether the set of limiting probabilities is at least dense in $[0, 1]$. As it turns out this is never the case (for $\mathcal{G}$ an addable, minor-closed class):

Proposition 1.5 *Let $\mathcal{G}$ be an addable, minor-closed class of graphs and let $G_n \in_u \mathcal{G}_n$. For every $\varphi \in \text{MSO}$ we have either $\lim_{n\to\infty} \mathbb{P}(G_n \models \varphi) \leq 1 - \mathrm{e}^{-1/2} \approx 0.3935$ or $\lim_{n\to\infty} \mathbb{P}(G_n \models \varphi) \geq \mathrm{e}^{-1/2} \approx 0.6065$.*

Next, one might ask for the topology of the set of limiting probabilities. Could they for instance form some strange, fractal-like set? (See the last section of this paper for an example of a model of random graphs, where such things do indeed happen.) The next theorem shows that the set of limiting probabilities is relatively well-behaved, and also that the limits of FO-sentences are dense in the set of limits of MSO-sentences. We denote by $\text{cl}(A)$ the topological closure of A in $\mathbb{R}$.

Theorem 1.6 *Let $\mathcal{G}$ be an addable, minor-closed class of graphs. Then*

$$\text{cl}\left(\left\{\lim_{n\to\infty} \mathbb{P}(G_n \models \varphi) : \varphi \in \text{MSO}\right\}\right) = \text{cl}\left(\left\{\lim_{n\to\infty} \mathbb{P}(G_n \models \varphi) : \varphi \in \text{FO}\right\}\right)$$

and this set is a finite union of intervals.

Let us remark that, in view of Proposition 1.5, the closure always consists of at least two disjoint intervals. In Section 6 we conjecture that there are at least four disjoint intervals for every addable, minor-closed class. Let us also observe that it follows from this last theorem that the closure of the FO-limiting probabilities for the random graph embeddable on S, for any surface S, coincides with the closure of the MSO-limiting probabilities for the class of planar graphs.

Perhaps surprisingly, the closure of the set of MSO-limiting probabilities can actually be determined exactly for two important examples. The first one is the class of forests.

Theorem 1.7 *If $\mathcal{G}$ is the class of forests and $G_n \in_u \mathcal{G}_n$ then*

$$\mathrm{cl}\left(\left\{\lim_{n\to\infty}\mathbb{P}(G_n \models \varphi) : \varphi \in \mathrm{MSO}\right\}\right)$$
$$= \left[0, 1-(1+\mathrm{e}^{-1})\mathrm{e}^{-1/2}\right] \cup \left[\mathrm{e}^{-3/2}, 1-\mathrm{e}^{1/2}\right] \cup \left[\mathrm{e}^{-1/2}, 1-\mathrm{e}^{-3/2}\right] \cup \left[(1+\mathrm{e}^{-1})\mathrm{e}^{-1/2}, 1\right].$$

The second class for which we can determine the closure of the set of limiting probabilities of MSO sentences is the class of planar graphs. To describe the results, we need the exponential generating function corresponding to a class of graphs $\mathcal{G}$, which is defined as $G(z) = \sum_{n=0}^{\infty} |\mathcal{G}_n| \frac{z^n}{n!}$.

Theorem 1.8 *If $\mathcal{G}$ is the class of planar graphs and $G_n \in_u \mathcal{G}_n$ then the set*

$$\mathrm{cl}\left(\left\{\lim_{n\to\infty}\mathbb{P}(G_n \models \varphi) : \varphi \in \mathrm{MSO}\right\}\right),$$

is the union of 108 disjoint intervals that all have exactly the same length, which is approximately $5 \cdot 10^{-6}$. The endpoints of these intervals are given explicitly in Theorem 4.8 below in terms of ρ and $G(\rho)$, where $G(z)$ is the exponential generating function for planar graphs, and ρ is its radius of convergence (see details in Section 4.4).

Until now we have dealt with addable minor-closed classes and graphs embeddable on a fixed surface, which in view of Theorems 1.3 and 1.4 (see also the results we list in Section 2.2) behave rather similarly to planar graphs. It is thus natural to ask to which extent our results can be expected to carry over to the non-addable case. Random graphs from several non-addable classes have recently been investigated in [7] and the results in that paper demonstrate that they can display behaviour very different from the addable case. In Section 5, we analyse three examples of non-addable graph classes from the logical limit laws point of view; and the results there are in stark contrast with the results on addable graph classes.

For $k \in \mathbb{N}$ a fixed integer, the collection $\mathcal{G}$ of all graphs whose every component has no more than k vertices is a minor-closed class that is not addable. Of course now $\mathcal{C}_n$, the set of connected graphs from $\mathcal{G}$ on n vertices, is empty for $n > k$. So it does not make sense to consider the random connected graph $C_n \in_u \mathcal{C}_n$. For the "general" random graph $G_n \in_u \mathcal{G}_n$ we see that, contrary to the addable case, every MSO sentence has a limiting probability that is either zero or one.

Theorem 1.9 *Let $k \in \mathbb{N}$ be fixed, let $\mathcal{G}$ be the class of all graphs whose components have at most k vertices, and let $G_n \in_u \mathcal{G}_n$. Then G_n obeys the* MSO*-zero-one law.*

Another simple example of a non-addable class of graphs $\mathcal{G}$ is formed by forests of paths (every component is a path). Note that now we can speak of $C_n \in_u \mathcal{C}_n$, the random path on n vertices, but it is still a rather uninteresting object as the only randomness is in the labels of the vertices. In Section 5.2 we give an MSO-sentence whose probability of holding for the random path C_n is zero for even n and one for odd n, disproving even the MSO-*convergence* law for the random path. On the other hand, we are able to prove the MSO-convergence law for the "general" random graph from $\mathcal{G}$. And, finally it turns out that now, contrary to the addable case, the limiting probabilities are in fact dense in $[0, 1]$.

Theorem 1.10 *Let C_n be the random path on n vertices, and let G_n be the random forest of paths on n vertices. Then the following hold:*

(i) *The* MSO*-convergence law fails for* C_n*;*

(ii) *The* MSO*-convergence law holds for* G_n*;*

(iii) $\operatorname{cl}\left(\left\{\lim_{n\to\infty}\mathbb{P}(G_n \models \varphi) : \varphi \in \mathrm{MSO}\right\}\right) = [0,1]$.

Of course graphs with bounded component size and forests of paths are relatively simple classes of graphs. We turn attention to a non-addable class of graphs that is slightly more challenging to analyze. A *caterpillar* consists of a path with vertices of degree one attached to it. A forest of caterpillars is a graph all of whose components are caterpillars. In Section 5.3 we construct an FO sentence such that the probability that the random caterpillar satisfies it converges, but not to zero or one. We are again able to show that the convergence law holds for the "general" random graph from the class (the random forest of caterpillars), but we are only able to do this if we restrict attention to FO logic – for what appears to be mainly technical reasons. And finally, also here the limiting probabilities (of FO sentences) turn out to be dense in $[0,1]$.

Theorem 1.11 *Let* C_n *be the random caterpillar on* n *vertices, and let* G_n *be the random forest of caterpillars on* n *vertices. Then the following hold:*

(i) *The* FO*-zero-one law fails for* C_n*;*

(ii) *The* FO*-convergence law holds for* G_n*;*

(iii) $\operatorname{cl}\left(\left\{\lim_{n\to\infty}\mathbb{P}(G_n \models \varphi) : \varphi \in \mathrm{FO}\right\}\right) = [0,1]$.

In the next section we give the notation and results from the literature that we need for the proofs of our main results. In Section 3 we give the proofs of Theorem 1.1 through 1.4. In Section 4 we prove Proposition 1.5 and Theorem 1.6 through 1.8. The proofs of our results on non-addable classes can be found in Section 5. Finally, in Section 6 we give some additional thoughts and open questions that arise from our work.

2 Notation and preliminaries

Throughout the paper, we write $[n] := \{1, \dots, n\}$. If G is a graph then we denote its vertex set by $V(G)$ and its edge set by $E(G)$. Its cardinalities are denoted by $v(G) := |V(G)|$ and $e(G) := |E(G)|$. For $u \in V(G)$, we denote by $B_G(u, r)$, called *r-neighbourhood of* u, the subgraph of G induced by the set of all vertices of graph distance at most r from u. When the graph is clear from the context we often simply write $B(u, r)$. Occasionally we write $x \sim y$ to denote that x is adjacent to y, in some graph clear from the context.

If G, H are graphs, then $G \cup H$ denotes the *vertex-disjoint union.* That is, we ensure that $V(G) \cap V(H) = \emptyset$ (by swapping H for an isomorphic copy if needed) and then we simply take $V(G \cup H) = V(G) \cup V(H), E(G \cup H) = E(G) \cup E(H)$. If n is an integer and G a graph then nG denotes the vertex-disjoint union of n copies of G.

An *unlabelled graph* is, formally, an isomorphism class of graphs. In this paper we deal with both labelled and unlabelled graphs. We will occasionally be a bit sloppy with the distinction between the two (for instance, by taking the vertex-disjoint union of a labelled and an unlabelled graph) but no confusion will arise. A class of graphs $\mathcal{G}$ always means a collection of graphs closed under isomorphism. We denote by $\mathcal{UG}$ the collection of unlabelled graphs corresponding to $\mathcal{G}$.

Following McDiarmid [30], we denote by $\mathrm{Big}(G)$ the largest component of G. In case of ties we take the lexicographically first among the components of the largest order (i.e., we look at the labels of the vertices and take the component in which the smallest label occurs). The 'fragment' $\mathrm{Frag}(G)$ of G is what remains after we remove $\mathrm{Big}(G)$.

Recall that a random variable is *discrete* if it takes values in some countable set Ω. For discrete random variables taking values in the same set Ω, the *total variation distance* is defined as:

$$\mathrm{dist}_{\mathrm{TV}}(X, Y) = \max_{A \subseteq \Omega} |\mathbb{P}(X \in A) - \mathbb{P}(Y \in A)|.$$

Alternatively, by some straightforward manipulations (see [26, Proposition 4.2]), we can write $\mathrm{dist}_{\mathrm{TV}}(X,Y) = \frac{1}{2}\sum_{a\in\Omega}|\mathbb{P}(X=a) - \mathbb{P}(Y=a)|$. If $X, X_1, X_2, \dots$ are discrete random variables, we say that X_n *tends to* X *in total variation* (notation: $X_n \to_{\mathrm{TV}} X$) if $\lim_{n\to\infty} \mathrm{dist}_{\mathrm{TV}}(X_n, X) = 0$.

Throughout this paper, $\mathrm{Po}(\mu)$ denotes the Poisson distribution with parameter μ. We make use of the following incarnation of the Chernoff bounds. A proof can for instance be found in Chapter 1 of [36].

Lemma 2.1 *Let $Z \stackrel{d}{=} \mathrm{Po}(\mu)$. Then the following bounds hold:*

- **(i)** *For all $k \geq \mu$ we have $\mathbb{P}(Z \geq k) \leq \mathrm{e}^{-\mu H(k/\mu)}$, and*
- **(ii)** *For all $k \leq \mu$ we have $\mathbb{P}(Z \leq k) \leq \mathrm{e}^{-\mu H(k/\mu)}$,*

where $H(x) := x\ln x - x + 1$. ■

In Section 4, we make use of a general result on the set of all sums of subsequences of a given summable sequence of nonnegative numbers. The following observation goes back a hundred years, to the work of Kakeya [22].

Lemma 2.2 ([22]) *Let $p_1, p_2, \dots$ be a summable sequence of nonnegative numbers. If $p_i \leq \sum_{j>i} p_j$ for every $i \in \mathbb{N}$, then*

$$\left\{\sum_{i\in A} p_i : A \subseteq \mathbb{N}\right\} = \left[0, \sum_{i=1}^{\infty} p_i\right].$$

From this last lemma it is relatively straightforward to derive the following observation – see for instance [33, Equation (3) and Proposition 6].

Corollary 2.3 *Let $p_1, p_2, \dots$ be a summable sequence of nonnegative numbers, and suppose there is an $i_0 \in \mathbb{N}$ such that $p_i \leq \sum_{j>i} p_j$ for all $i > i_0$. Then the set of sums of subsequences of $(p_n)_n$ is a union of 2^{i_0} translates of the interval $\left[0, \sum_{i>i_0} p_i\right]$, namely:*

$$\left\{\sum_{i\in A} p_i : A \subseteq \mathbb{N}\right\} = \bigcup_{x_1,\dots,x_{i_0}\in\{0,1\}} \left[x_1p_1 + \dots + x_{i_0}p_{i_0}, x_1p_1 + \dots + x_{i_0}p_{i_0} + \sum_{i>i_0} p_i\right]. \tag{1}$$

2.1 Logical preliminaries

A variable x in a logical formula is called *bound* if it has a quantifier. Otherwise, it is called *free*. A *sentence* is a formula without free variables.

The *quantifier depth* $\mathrm{qd}(\varphi)$ of an MSO formula φ is, informally speaking, the longest chain of 'nestings' of quantifiers. More formally, it is defined inductively using the axioms **1)** $\mathrm{qd}(\neg\varphi) = \mathrm{qd}(\varphi)$, **2)** $\mathrm{qd}(\varphi \vee \psi) = \mathrm{qd}(\varphi \wedge \psi) = \mathrm{qd}(\varphi \Rightarrow \psi) = \max(\mathrm{qd}(\varphi), \mathrm{qd}(\psi))$, **3)** $\mathrm{qd}(\exists x : \varphi) = \mathrm{qd}(\forall x : \varphi) = \mathrm{qd}(\exists X : \varphi) = \mathrm{qd}(\forall X : \varphi) = 1 + \mathrm{qd}(\varphi)$, **4)** $\mathrm{qd}(x = y) = \mathrm{qd}(x \sim y) = \mathrm{qd}(x \in X) = 0$.

For two graphs G and H, the notation $G \equiv_k^{\mathrm{MSO}} H$ denotes that every $\varphi \in \mathrm{MSO}$ with $\mathrm{qd}(\varphi) \leq k$ is either satisfied by both G and H or false in both. We define $G \equiv_k^{\mathrm{FO}} H$ similarly. It is immediate from the definition that $\equiv_k^{\mathrm{MSO}}, \equiv_k^{\mathrm{FO}}$ are both equivalence relations on the set of all graphs. What is more, for every k there are only finitely many equivalence classes (for a proof see, e.g., [13, Proposition 3.1.3]):

Lemma 2.4 *For every $k \in \mathbb{N}$, the relation $\equiv_k^{\mathrm{MSO}}$ has finitely many equivalence classes. The same holds for $\equiv_k^{\mathrm{FO}}$.*

The MSO-*Ehrenfeucht-Fraïssé-game* $\mathrm{EHR}_k^{\mathrm{MSO}}(G,H)$ is a two-player game played on two graphs G, H for k rounds. The game is played as follows. There are two players, Spoiler and Duplicator. In each round $1 \le i \le k$, Spoiler is first to move, selects one of the two graphs G or H (and in particular is allowed to switch the graph at any new round), and does either a vertex-move or a set-move. That is, Spoiler either selects a single vertex or a subset of the vertices. If Spoiler did a vertex-move then Duplicator now has to select a vertex from the graph that Spoiler did not play on, and otherwise Duplicator has to select a subset of the vertices from the graph Spoiler did not play on. After k rounds the game is finished. To decide on the winner, we first need to introduce some additional notation. Let $I \subseteq \{1,\dots,k\}$ be those rounds in which a vertex-move occurred. For $i \in I$, let $x_i \in V(G), y_i \in V(H)$ be the vertices that were selected in round i. For $i \notin I$, let $X_i \subseteq V(G), Y_i \subseteq V(H)$ be the subsets of vertices that were selected in round i. Duplicator has won if the following three conditions are met:

1) $x_i = x_j$ if and only if $y_i = y_j$, for all $i, j \in I$, and;

2) $x_i x_j \in E(G)$ if and only if $y_i y_j \in E(H)$, for all $i, j \in I$, and;

3) $x_i \in X_j$ if and only if $y_i \in Y_j$, for all $i \in I, j \notin I$.

Otherwise Spoiler has won. We say that $\mathrm{EHR}_k^{\mathrm{MSO}}(G,H)$ *is a win for Duplicator* if there exists a winning strategy for Duplicator. (I.e., no matter how Spoiler plays, Duplicator can always respond so as to win in the end.) The FO-Ehrenfeucht-Fraïssé-game $\mathrm{EHR}_k^{\mathrm{FO}}(G,H)$ is defined just like $\mathrm{EHR}_k^{\mathrm{MSO}}(G,H)$, except that set-moves do not exist in that game.

The following lemma shows the relation between these games and logic. A proof can for instance be found in [13, Theorems 2.2.8 and 3.1.1].

Lemma 2.5 *$G \equiv_k^{\mathrm{MSO}} H$ if and only if $\mathrm{EHR}_k^{\mathrm{MSO}}(G,H)$ is a win for Duplicator. Similarly, $G \equiv_k^{\mathrm{FO}} H$ if and only if $\mathrm{EHR}_k^{\mathrm{FO}}(G,H)$ is a win for Duplicator.*

The Ehrenfeucht-Fraïssé-game is a convenient tool for proving statements about logical (in-) equivalence. It can be used for instance to prove that the statement 'G is connected' cannot be expressed as an FO-sentence, and that similarly 'G has a Hamilton cycle' cannot be expressed by an MSO-sentence. (See for instance [38, Theorem 2.4.1].)

The following two standard facts about $\equiv_k^{\mathrm{MSO}}$ are essential tools in our arguments. They can for instance be found in [11, Theorems 2.2 and 2.3], where they are proved in a greater level of generality.

Lemma 2.6 *If $H_1 \equiv_k^{\mathrm{MSO}} G_1$ and $H_2 \equiv_k^{\mathrm{MSO}} G_2$ then $H_1 \cup H_2 \equiv_k^{\mathrm{MSO}} G_1 \cup G_2$. The same conclusion holds w.r.t. $\equiv_k^{\mathrm{FO}}$.*

Lemma 2.7 *For every $k \in \mathbb{N}$ there is an $a = a(k)$ such that the following holds. For every graph G and every $b \ge a$ we have $aG \equiv_k^{\mathrm{MSO}} bG$.*

Let us observe that the statement $\mathrm{dist}(x,y) \le r$, where dist denotes the graph distance, can easily be written as a first order formula whose only free variables are x, y. If φ is a first order formula then we denote by $\varphi^{B(x,r)}$ the formula in which all bound variables are 'relativised to $B(x,r)$'. This means that in $\varphi^{B(x,r)}$ all variables range over $B(x,r)$ only. (This can be achieved by inductively applying the substitutions $[\forall y : \psi(y)]^{B(x,r)} := \forall y : ((\mathrm{dist}(x,y) \le r) \Rightarrow \psi^{B(x,r)}(y))$ and $[\exists y : \psi(y)]^{B(x,r)} := \exists y : (\mathrm{dist}(x,y) \le r) \wedge \psi^{B(x,r)}(y)$.) A *basic local* sentence is a sentence of the form

$$\exists x_1,\dots,x_n : \left(\bigwedge_{1 \le i \le n} \psi^{B(x_i,\ell)}(x_i) \right) \wedge \left(\bigwedge_{1 \le i < j \le n} \mathrm{dist}(x_i,x_j) > 2\ell \right), \tag{2}$$

where $\psi(x)$ is a FO-formula whose only free variable is x. A *local sentence* is a boolean combination of basic local sentences. The following theorem captures the intuition that first order sentences in some sense can only capture local properties. It will help us to shorten some proofs in the sequel. Besides in [16], a proof can for instance be found in [13, Section 2.5].

Theorem 2.8 (Gaifman's theorem, [16]) *Every first order sentence is logically equivalent to a local sentence.*

2.2 Preliminaries on minor-closed classes

In this section we introduce some notions and results on minor-closed classes of graphs that we need in later arguments. We say that a class of graphs $\mathcal{G}$ is *decomposable* if $G \in \mathcal{G}$ if and only if every component of G is in $\mathcal{G}$. We say that $\mathcal{G}$ is *addable* if it is decomposable, and closed under adding an edge between vertices in distinct components. Let us mention (although we shall not use this anywhere in the paper) that a minor-closed class is addable if and only if it can be characterised by a list of excluded minors that are all 2-connected [30, p. 1].

Throughout this paper, $\mathcal{G}$ denotes a minor-closed class of graphs, $\mathcal{C}$ the *connected* graphs in $\mathcal{G}$, $\mathcal{G}_n$ the graphs of $\mathcal{G}$ on vertex set $\{1, \dots, n\}$, $\mathcal{C}_n$ the connected elements of $\mathcal{G}_n$, and $\mathcal{UG}$ denotes the unlabelled class corresponding to $\mathcal{G}$, i.e., the set of all isomorphism classes of graphs in $\mathcal{G}$. We define $\mathcal{UC}$,$\mathcal{UG}_n$,$\mathcal{UC}_n$ similarly. The *exponential generating function of* $\mathcal{G}$ is defined by

$$G(z) := \sum_{n=0}^{\infty} |\mathcal{G}_n| \frac{z^n}{n!},$$

and similarly $C(z) = \sum_{n=1}^{\infty} |\mathcal{C}_n| \frac{z^n}{n!}$. (Note that by convention the "empty graph" is not considered connected, so that $|\mathcal{G}_0| = 1$ and $|\mathcal{C}_0| = 0$.) If $\mathcal{G}$ is decomposable then it can be seen that $G(z)$ and $C(z)$ are related by the *exponential formula* (see [10, Lemma 2.1 (i)] or [15, Chapter II]).

$$G(z) = \exp(C(z)). \tag{3}$$

The *radius of convergence* of $G(z)$ will always be denoted by ρ. Note that we have

$$\rho = \left(\limsup_{n\to\infty} \left(\frac{|\mathcal{G}_n|}{n!} \right)^{1/n} \right)^{-1}. \tag{4}$$

By a result of Norine, Seymour, Thomas and Wollan [34], we know that $\rho > 0$ for every minor-closed class other than the class of all graphs. More detailed information on which values ρ can assume for minor-closed classes was obtained by Bernardi, Noy and Welsh [5]. Amongst other things, they showed that the radius of convergence is infinite if and only if $\mathcal{G}$ does not contain every path; and that otherwise, if $\mathcal{G}$ contains all paths, then $\rho \leq 1$. A class $\mathcal{G}$ is said to be *smooth* if

$$\lim_{n\to\infty} \frac{n|\mathcal{G}_{n-1}|}{|\mathcal{G}_n|} \qquad \text{exists and is finite.} \tag{5}$$

(In case this limit does exist then it must in fact equal ρ.) Smoothness turns out to be a key property in many proofs on enumerative and probabilistic aspects of graphs from minor-closed classes. The following result was proved by McDiarmid in [30]. The statement below combines Theorem 1.2 and Lemma 2.4 of [30].

Theorem 2.9 ([30]) *Let $\mathcal{G}$ be an addable, minor-closed class, and let $\mathcal{C} \subseteq \mathcal{G}$ be the corresponding class of connected graphs. Then $\mathcal{C}$ and $\mathcal{G}$ are both smooth.*

A crucial object in the literature on random graphs from minor closed classes is the Boltzmann-Poisson random graph, which we define next.

Definition 2.10 (Boltzmann–Poisson random graph) *Let $\mathcal{G}$ be a decomposable class of graphs, and let ρ be the radius of convergence of its exponential generating function $G(z)$. If $G(\rho) < \infty$ then the* Boltzmann–Poisson random graph *corresponding to $\mathcal{G}$ is the unlabelled random graph R satisfying:*

$$\mathbb{P}(R = H) = \frac{1}{G(\rho)} \cdot \frac{\rho^{v(H)}}{\operatorname{aut}(H)} \qquad \textit{for all } H \in \mathcal{UG}. \tag{6}$$

Theorem 1.3 of [30] establishes that this indeed defines a probability distribution taking values in $\mathcal{UG}$. The same paper also establishes the following result, which state as a separate lemma for future convenience.

Lemma 2.11 *Let $\mathcal{G}, \rho$ and R be as in Definition 2.10. Let $H_1, \dots, H_k \in \mathcal{G}$ be non-isomorphic connected graphs from $\mathcal{G}$ and let Z_i denote the number of components of R that are isomorphic to H_i. Then $Z_1, \dots, Z_k$ are independent Poisson random variables with means $\mathbb{E}Z_i = \rho^{v(H_i)}/\operatorname{aut}(H_i)$.*

The following is a slight rewording of Theorem 1.5 in [30], where something stronger is proved.

Theorem 2.12 ([30]) *Let $\mathcal{G}$ be an addable minor-closed class other than the class of all graphs, let ρ be its radius of convergence and let $G_n \in_u \mathcal{G}_n$. Then $G(\rho) < \infty$ and if F_n denotes the isomorphism class of* $\operatorname{Frag}(G_n)$ *(so F_n is the* unlabelled *version of* $\operatorname{Frag}(G_n)$*) then $F_n \to_{TV} R$, where R is the Boltzmann–Poisson random graph associated with $\mathcal{G}$.*

This powerful result has several useful immediate corollaries, as pointed out in [30]. For instance, it follows that, if $\mathcal{G}$ is addable and minor-closed, then $|\operatorname{Big}(G_n)| = n - O(1)$ w.h.p., and

$$\lim_{n\to\infty} \mathbb{P}(G_n \text{ is connected }) = \lim_{n\to\infty} \mathbb{P}(\operatorname{Frag}(G_n) = \emptyset) = \mathbb{P}(R = \emptyset) = \frac{1}{G(\rho)}. \tag{7}$$

McDiarmid, Steger and Welsh [31] remarked that for the case of forests, the asymptotic probability of being connected is $1/G(\rho) = \mathrm{e}^{-1/2}$, and they also conjectured that this is the smallest possible value over all weakly addable graph classes (a class of graphs $\mathcal{G}$ is weakly addable if adding an edge between distinct component of a graph in $\mathcal{G}$ always produces another graph in $\mathcal{G}$). This conjecture is still open in general, but it was recently proved under some conditions which are met by addable, minor closed classes, by two independent teams: Addario-Berry, McDiarmid and Reed [1], and Kang and Panagiotou [23, Theorem 1.1]. A corollary of their result is the following.

Theorem 2.13 ([1, 23]) *If $\mathcal{G}$ is an addable, minor-closed class of graphs then $G(\rho) \le \sqrt{\mathrm{e}}$.*

Let H be a connected graph with a distinguished vertex r, the 'root'. We say that G contains a *pendant copy of H* if G contains an induced subgraph isomorphic to H, and there is exactly one edge between this copy of H and the rest of the graph, and this edge is incident with the root r. McDiarmid [30, Theorem 1.7] proved the following remarkable result:

Theorem 2.14 ([30]) *Let $\mathcal{G}$ be an addable, minor-closed class and $G_n \in_u \mathcal{G}_n$. Let $H \in \mathcal{G}$ be any fixed, connected (rooted) graph. Then, w.h.p., G_n contains linearly many pendant copies of H.*

While not explicitly remarked in [30], the result carries over to the random connected graph from $\mathcal{G}$.

Corollary 2.15 *Let $\mathcal{G}$ be an addable, minor-closed class and let $C_n \in_u \mathcal{C}_n$ be the random* connected *graph from $\mathcal{G}$. If $H \in \mathcal{G}$ is any fixed, connected (rooted) graph then, w.h.p., C_n contains linearly many pendant copies of H.*

Proof: By Theorem 2.14, there is a constant $\alpha > 0$ such that $\mathbb{P}(E_n) = 1 - o(1)$, where E_n denotes the event that G_n contains at least αn pendant copies of H. Let F_n denote the event that C_n contains at least αn pendant copies of H, and let A_n denote the event that G_n is connected. Aiming for a contradiction, let us suppose that $\liminf_{n\to\infty} \mathbb{P}(F_n) = \beta < 1$. Observe that if we condition on A_n, we find that G_n is distributed like C_n (G_n is now chosen uniformly at random from all connected graphs from $\mathcal{G}_n$). This implies that

$$\begin{aligned}
\liminf_{n\to\infty} \mathbb{P}(E_n) &= \liminf_{n\to\infty} \left[\mathbb{P}(F_n) \cdot \mathbb{P}(A_n) + \mathbb{P}(E_n | A_n^c) \cdot (1 - \mathbb{P}(A_n))\right] \\
&\le \liminf_{n\to\infty} \left[\mathbb{P}(F_n) \cdot \mathbb{P}(A_n) + 1 \cdot (1 - \mathbb{P}(A_n))\right] \\
&= \beta \cdot \tfrac{1}{G(\rho)} + (1 - \tfrac{1}{G(\rho)}),
\end{aligned}$$

using (7) for the last line. But this last expression is < 1, a contradiction. Hence we must have $\mathbb{P}(F_n) = 1 - o(1)$, as required. ■

In the paper [29], McDiarmid proved a result analogous to Theorem 2.12 above for the class $\mathcal{G}_S$ of all graphs embeddable on some fixed surface S under the additional assumption that $\mathcal{G}_S$ is smooth. That $\mathcal{G}_S$ *is* indeed smooth for every surface S was later established by Bender, Canfield and Richmond [2]. See also [9], where more detailed asymptotic information is derived for the number of graphs on n vertices from $\mathcal{G}_S$. By combining [29, Theorem 3.3] with [2, Theorem 2] we obtain:

Theorem 2.16 ([29, 2]) *Let S be any surface, let $\mathcal{G}$ be the class of all graphs embeddable on S and let $G_n \in_u \mathcal{G}_n$. If F_n denotes the isomorphism class of* $\mathrm{Frag}(G_n)$ *then $F_n \to_{TV} R$, where R is the Boltzmann–Poisson random graph associated with the class of* planar *graphs $\mathcal{P}$.*

Let us stress that the fragment in this last case follows the Boltzmann–Poisson distribution associated with the class of *planar* graphs. Hence the asymptotic distribution of the fragment is independent of the choice of surface S. Of course, the remarks following Theorem 2.12 also apply to the case of graphs on surfaces, where $\rho, G(\rho)$ are the values for the class of planar graphs.

Without having to assume smoothness, McDiarmid [29] was able to prove the analogue of Theorem 2.14 for graphs on surfaces.

Theorem 2.17 ([29]) *Let S be a fixed surface, let $\mathcal{G}$ be the class of all graphs embeddable on S and let and $G_n \in_u \mathcal{G}_n$. Let H be any fixed, connected (rooted)* planar *graph. Then, w.h.p., G_n contains linearly many pendant copies of H.*

A verbatim repeat of the proof of Corollary 2.15 now also yields:

Corollary 2.18 *Let S be a fixed surface, let $\mathcal{G}$ be the class of all graphs embeddable on S and let and $C_n \in_u \mathcal{C}_n$ be the random* connected *graph from $\mathcal{G}$. Let H be any fixed, connected (rooted)* planar *graph. Then, w.h.p., C_n contains linearly many pendant copies of H.*

We also need another powerful result showing that the random graph embeddable on a fixed surface is locally planar in the sense given by the next theorem. It was essentially proved in [9], but not stated there explicitly. For this reason we give a short sketch of how to extract a proof from the results in [9].

Theorem 2.19 *Let S be any fixed surface, let $\mathcal{G}$ be the class of all graphs embeddable on S, let $G_n \in_u \mathcal{G}_n$, and let $r \in \mathbb{N}$ be fixed. Then w.h.p. $B_{G_n}(v, r)$ is planar for all $v \in V(G_n)$.*

Proof sketch: Let M be an embedding of a graph G on a surface S. The face-width $\mathrm{fw}(M)$ of M is the minimum number of intersections of M with a simple non-contractible curve C on S. It is easy to see that this minimum is achieved when C meets M only at vertices of G. Notice that if $\mathrm{fw}(M) \geq 2r$, then all the balls in G of radius r are planar.

Fix a surface S. From the results in [9] it follows that for any fixed k, a random graph that can be embedded in S has an embedding in S with face-width at least k w.h.p. This is first established for 3-connected graphs embeddable in S; see [9, Lemma 4.2]. It is also proved that w.h.p. a random connected graph G embeddable in S has a unique 3-connected component T of linear size (whose genus is the genus of S), and the remaining 3- and 2-connected components are planar. The component T is not uniform among all 3-connected graphs with the same number of vertices, since it carries a weight on the edges. But since Lemma 4.2 in [9] holds for weighted graphs, the component M has large face-width w.h.p. Since the remaining components are planar, this also applies to G. By Theorem 2.14, this implies the same result for arbitrary graphs. Analogous results were obtained independently in [4]. ■

Again a nearly verbatim repeat of the proof of Corollary 2.15 shows:

Corollary 2.20 *Let S be any surface, let $\mathcal{G}$ be the class of all graphs embeddable on S, let $C_n \in_u \mathcal{C}_n$ be the random connected graph from $\mathcal{G}$, and let $r \in \mathbb{N}$ be fixed. Then w.h.p. $B_{C_n}(v, r)$ is planar for all $v \in V(C_n)$.*

As evidenced by the results we have listed here, random graphs embeddable on a fixed surface behave rather similarly to random planar graphs. In particular, despite not being an addable class, the size of their largest component essentially behaves like the largest component of a random planar graph (as the number of vertices not in the largest component is described by the Boltzmann-Poisson random graph for planar graphs – cf. Theorem 2.16).

In general, however, non-addable minor-closed graph classes can display a very different behaviour: see, for example, the recent preprint [7], where several non-addable graph classes are analysed in detail. In particular, the largest component can happen to be sublinear w.h.p., as opposed to $n - O(1)$ w.h.p. for the special non-addable class of graphs on a fixed surface.

Therefore, we cannot expect a result like Theorem 2.12 to hold for general smooth, decomposable, minor-closed classes. Using another result of McDiarmid, we are however able to recover a Poisson law for component counts under relatively general conditions. The following lemma is a special case of Lemma 4.2 in [30].

Lemma 2.21 *Let $\mathcal{G}$ be a smooth, decomposable, minor-closed class of graphs and let $G_n \in_u \mathcal{G}_n$. Let $H_1, \ldots, H_k \in \mathcal{G}$ be non-isomorphic, fixed, connected graphs, and let N_i denote the number of components of G_n isomorphic to H_i. Then*

$$(N_1, \ldots, N_k) \to_{TV} (Z_1, \ldots, Z_k),$$

where the Z_i are independent Poisson random variables with means $\mathbb{E}Z_i = \rho^{v(H_i)}/\operatorname{aut}(H_i)$, and ρ is the radius of convergence of the exponential generating function for $\mathcal{G}$. ■

Two examples of minor-closed classes that are decomposable, but not addable, are forests of paths and forests of caterpillars. Very recently Bousquet-Mélou and Weller [7] derived precise asymptotics for the numbers of labelled forests of paths (resp. caterpillars). As a direct corollary of their Propositions 23 and 26 we have:

Theorem 2.22 ([7]) *The classes $\{$forests of paths$\}$ and $\{$forests of caterpillars$\}$ are both smooth.*

Bousquet-Mélou and Weller [7] also analysed the class of all graphs whose components have order at most k (fixed). This is clearly a minor-closed class that is decomposable, but not addable. It is in fact not smooth, but a similar property follows from Proposition 20 in [7].

Corollary 2.23 ([7]) *Let $k \in \mathbb{N}$ be fixed and let $\mathcal{G}$ be the class of all graphs whose components have at most k vertices. There is a constant $c = c(k)$ such that*

$$\frac{n|\mathcal{G}_{n-1}|}{|\mathcal{G}_n|} \sim c \cdot n^{1/k}.$$

3 The logical limit laws for the addable and surface case

3.1 The MSO-zero-one law for addable classes

The main logical ingredient we need is the following theorem that is inspired by a construction of McColm [28].

Theorem 3.1 *Let $\mathcal{G}$ be an addable, minor-closed class of graphs. For every $k \in \mathbb{N}$, there exists a connected (rooted) graph $M_k \in \mathcal{G}$ with the following property. For every connected $G \in \mathcal{G}$ that contains a pendant copy of M_k, it holds that $G \equiv_k^{\mathrm{MSO}} M_k$.*

Before starting the proof we need to introduce some more notation. Recall that a rooted graph is a graph G with a distinguished vertex $r \in V(G)$. If G, H are two rooted graphs then we say that a third graph I is the result of *identifying their roots* if I can be obtained as follows. Without loss of generality we can assume $V(G) = \{r\} \cup A, V(H) = \{r\} \cup B$ where r is the root in both graphs and A, B are disjoint. Then $I = (V(G) \cup V(H), E(G) \cup E(H))$ is the graph we get by 'identifying the roots'.

The *rooted Ehrenfeucht-Fraïssé-game* $\mathrm{EHR}_k^{\mathrm{rMSO}}(G, H)$ is played on two rooted graphs G, H with roots r_G and r_H in the same way as the unrooted version. The only difference is that for Duplicator to win, the following additional conditions have to be met in addition to conditions **1)**, **2)**, **3)** from the description of the Ehrenfeucht–Fraïssé-game in Section 2.1: **4)** $x_i = r_G$ if and only if $y_i = r_H$, **5)** $x_i \sim r_G$ if and only if $y_i \sim r_H$, and **6)** $r_G \in X_i$ if and only if $r_H \in Y_i$. We can view it as the ordinary game with one additional move, where the first move of both players is predetermined to be a vertex-move selecting the root. We write $G \equiv_k^{\mathrm{rMSO}} H$ if $\mathrm{EHR}_k^{\mathrm{rMSO}}(G, H)$ is a win for Duplicator. Note that $\equiv_k^{\mathrm{rMSO}}$ is an equivalence relation with finitely many equivalence classes (using that $\equiv_{k+1}^{\mathrm{MSO}}$ has finitely many equivalence classes and the previous remark).

The next two lemmas are the natural analogues of Lemmas 2.6 and 2.7 for the rooted MSO-Ehrenfeucht-Fraïssé-game. For completeness we include self-contained proofs.

Lemma 3.2 *Suppose that $G_1 \equiv_k^{\mathrm{rMSO}} H_1, G_2 \equiv^{\mathrm{rMSO}} H_2$, and let G be obtained by identifying the roots of G_1, G_2 and let H be obtained by identifying the roots of H_1, H_2. Then $G \equiv_k^{\mathrm{rMSO}} H$.*

Proof: It is convenient to assume that $V(G_1)$ and $V(G_2)$ have exactly one element in common, the root r_G of G, and to identify G_1, G_2 with the copies in G. And similarly for H_1, H_2 and H.

The winning strategy for Duplicator is as follows. If Spoiler does a vertex move, say he selects v of G_ℓ with $\ell \in \{1, 2\}$, then Duplicator responds by selecting a vertex of H_ℓ according to his winning strategy for $\mathrm{EHR}_k^{\mathrm{rMSO}}(G_\ell, H_\ell)$. Note that no confusion can arise if Spoiler selects the root of either graph since Duplicator must then select the root of the other graph (otherwise he loses immediately). Similarly, Duplicator never selects the root if Spoiler did not also select the root.

If Spoiler does a set move then Duplicator responds as follows. Suppose Spoiler selected $X \subseteq V(G)$, and let us write $X_\ell := X \cap V(G_\ell)$ for $\ell = 1, 2$. Then Duplicator selects a set $Y_\ell \subseteq V(H_\ell)$ for each $\ell \in \{1, 2\}$, according to the winning strategy for $\mathrm{EHR}_k^{\mathrm{rMSO}}(G_\ell, H_\ell)$, and then sets $Y := Y_1 \cup Y_2$ as his response to Spoiler's move. Again, no confusion can arise because of the presence or not of the root in X. If Spoiler selects a subset $Y \subseteq V(H)$ then Duplicator responds analogously. This is a winning strategy for Duplicator as every edge of G is either an edge of G_1 or of G_2 and every vertex of G other than the root is either a vertex of G_1 or of G_2; and similarly for H. ■

Lemma 3.3 *For every $k \in \mathbb{N}$ there is an $a = a(k)$ such that the following holds. For every rooted graph G and every $b \geq a$, if A is obtained from a copies of G and identifying the roots, and B is obtained from b copies of G and identifying the roots, then $A \equiv_k^{\mathrm{rMSO}} B$.*

Proof: Before proving the full statement, we prove a seemingly weaker statement.

Claim. Let G and $a \geq 2^{k \cdot v(G)}$ be arbitrary, and let a rooted graph A be obtained by identifying the roots of $a+1$ copies of G and a rooted graph B by identifying a copies of G. Then $A \equiv_k^{\mathrm{rMSO}} B$.

Proof of Claim: To prove the claim, let $A_1, \dots, A_{a+1}$ be the copies of G that make up A, and let $B_1, \dots, B_a$ be the copies of G that make up B. For any i, j and $v \in V(G)$ we denote by v_i^A (resp. v_j^B) the unique copy of vertex v inside the copy A_i of G (resp. inside the copy B_j of G).

Note that because of the demands **4)** and **5)** for Duplicator's win, Spoiler basically 'wastes a move' when selecting the root of either graph in a vertex-move, since the root already behaves like a marked vertex. So if Spoiler can win $\mathrm{EHR}_k^{\mathrm{rMSO}}(A, B)$, then winning is possible without ever making a vertex-move selecting the root of either graph. In the sequel we thus assume that Spoiler's vertex-moves never selects the root.

We describe the situation after any move of the game $\mathrm{EHR}_k^{\mathrm{rMSO}}(A,B)$ by the graphs A,B, together with some 'vertex-move-marks' and 'set-move-marks' on their vertices, which record the number of the move when the respective vertex or vertex-set was selected, and whether it was a vertex- or set-move. (The information whether it was Spoiler or Duplicator who made a choice is not recorded.)

For any move $0 \le k' \le k$, we say that A_i and B_j are *marked identically* to each other (after move k') if for every move $k'' \le k'$ the following holds: if k'' was a vertex-move then v_i^A was marked if and only if v_j^B was marked (for all $v \in V(G)$) and if k'' was a set-move then v_i^A was in the selected subset of $V(A)$ if and only if v_j^B was in the selected subset of $V(B)$ (for all $v \in V(G)$). Similarly, we also speak of A_i being marked identically to A_j if and only if no vertex-move occurred on either one, and every set-move until now selected the same subset from both.

For any $0 \le k' \le k$, let us say that the situation of the game after move k' is *good* (for Duplicator) if the following holds:

- A_i and B_i are marked identically for all $1 \le i \le a$;
- no vertex of A_{a+1} was marked by a vertex-move until now;
- there are at least $2^{(k-k')\cdot v(G)}$ indices $1 \le i \le a$ such that B_i is marked identically to A_{a+1}.

Observe that if after move $k' = k$ the situation is still good, Duplicator has won the game. Our aim is to show by induction that Duplicator can indeed achieve this situation, up to a relabelling of the indices. Clearly, the situation is good at the beginning of the game, when no moves have been played yet, corresponding to $k = 0$.

Now assume that after move $k' < k$, the situation is good. We show that, no matter what Spoiler does in move $k' + 1$, Duplicator can respond in such a way that after move $k' + 1$ the situation is still good, possibly after some relabelling.

To see this, let us first suppose that Spoiler does a vertex-move: if this marks the vertex $v_i^A \in V(A_i)$ for some $v \in V(G)$ and $1 \le i \le a$, then Duplicator simply responds by marking the vertex $v_i^B \in V(B_i)$. Observe that now we are still in a good situation: A_i and B_i are marked identically for $i = 1,\dots,a$, and A_{a+1} has none of its vertices marked by a vertex-move and is marked identically to at least $2^{(k-k')\cdot v(G)} - 1 \ge 2^{(k-k'-1)\cdot v(G)}$ of the B_i. Similarly, if Spoiler chooses to mark the vertex $v_i^B \in V(B_i)$ for some $v \in V(G)$ and $1 \le i \le a$, then Duplicator can again respond by marking v_i^A and we are still good. Assume thus that Spoiler marked a vertex of A_{a+1}. Since there are $2^{(k-k')\cdot v(G)} \ge 1$ indices i such that A_i is marked identically to A_{a+1} we can just swap indices and arrive at the situation where Spoiler still only chose a vertex in an A_i with $i \le a$. But then the induction is again done, by the above. This completes the case when Spoiler does a vertex-move.

We now consider set-moves. In the rest of the proof, let $I \subseteq [a]$ be the set of those indices $i \in [a]$ for which B_i is marked identically to A_{a+1}, after move k'.

First suppose that Spoiler in move $k' + 1$ selected a subset $X \subseteq V(A)$. Observe that, since G has $2^{v(G)}$ subsets in total, the set $I \cup \{a + 1\}$ is partitioned into $L \le 2^{v(G)}$ subsets $I_1,\dots,I_L$ such that, for any $1 \le \ell \le L$, if $i,j \in I_\ell$ then A_i, A_j are marked identically after Spoiler's move. There must be some ℓ such that $|I_\ell| \ge (|I| + 1)/2^{v(G)} > 2^{(k-k'-1)\cdot v(G)}$. Relabelling if necessary, we can assume without loss of generality that $a + 1 \in I_\ell$ for such an index ℓ. (So in particular A_{a+1} is marked identically to at least $2^{(k-k'-1)\cdot v(G)}$ of the other A_i.) Duplicators response is simply to select $Y \subseteq V(B)$ according to the rule that $v_i^B \in Y$ if and only if $v_i^A \in X$ (for all $1 \le i \le a$ and all $v \in V(G)$). Note that no confusion can arise because of the root. It is easily seen that this way, the situation is still good after move $k' + 1$.

Suppose then that Spoiler selected a subset $Y \subseteq V(B)$. Now I is partitioned into $L \le 2^{v(G)}$ subsets $I_1,\dots,I_L$ such that, for any $1 \le \ell \le L$, the sets B_i, B_j are marked identically after Spoiler's move whenever $i,j \in I_\ell$. There is some $1 \le \ell \le L$ such that $|I_\ell| \ge |I|/2^{v(G)} \ge 2^{(k-k'-1)\cdot v(G)}$. We fix such an ℓ and an $i_0 \in I_\ell$. Duplicators response is to select $X \subseteq V(A)$ according to the rules that $v_i^A \in X$ if and only if $v_i^B \in Y$ (for all $1 \leq i \leq a$ and all $v \in V(G)$) and that $v_{a+1}^A \in X$ if and only if $v_{i_0}^B \in Y$ (for all $v \in V(G)$). Again it is easily seen that the situation is still good after move $k' + 1$.

We have seen that indeed, no matter which move Spoiler chooses to make, Duplicator can always respond in such a way that the situation will stay good. Hence $A \equiv_k^{\mathrm{rMSO}} B$, which completes the proof of the claim. ■

Having proved the claim, we are ready for finish the proof of the lemma. Observe that, by repeated applications of the claim, we also have that $A \equiv_k^{\mathrm{rMSO}} B$ for all $a, b \geq 2^{k \cdot v(G)}$ if A is obtained by identifying the roots of a copies of G and B by identifying the roots of b copies.

Since there are finitely many equivalence classes for $\equiv_k^{\mathrm{rMSO}}$, there is a finite list of graphs $H_1, \dots, H_\ell$ such that every graph is equivalent to one of them. Let us set

$$a = a(k) := \max\left(2^{k \cdot v(H_1)}, \dots, 2^{k \cdot v(H_\ell)}\right).$$

Let G be an arbitrary graph, and $b \geq a$ be arbitrary. There is some $1 \leq i \leq \ell$ such that $G \equiv_k^{\mathrm{rMSO}} H_i$. Let A, A' be obtained by identifying the roots of a copies of G resp. H_i, and let B, B' be obtained by identifying the roots of b copies of G resp. H_i. By Lemma 3.2, we also have $A \equiv_k^{\mathrm{rMSO}} A'$ and $B \equiv_k^{\mathrm{rMSO}} B'$. By the claim and the observation we made immediately after its proof, we have $A' \equiv_k^{\mathrm{rMSO}} B'$. Hence also $A \equiv_k^{\mathrm{rMSO}} B$. This proves that our choice of $a(k)$ indeed works for *every* graph G, and concludes the proof of Lemma 3.3. ■

Proof of Theorem 3.1: The construction of M_k is as follows. Recall that $\mathcal{C}$ denotes the set of all connected elements of $\mathcal{G}$. Let r$\mathcal{G}$ be the set of all rooted graphs corresponding to $\mathcal{G}$. That is, for each element $G \in \mathcal{G}$ there are $v(G)$ elements in r$\mathcal{G}$, one for each choice of the root. We define r$\mathcal{C}$ similarly. As remarked previously, Lemma 2.4, despite being about $\equiv_k^{\mathrm{MSO}}$, implies that the relation $\equiv_k^{\mathrm{rMSO}}$ partitions the set of all rooted finite graphs, and hence in particular r$\mathcal{C}$, into finitely many equivalence classes. Hence there exists a finite set of *connected* rooted graphs $G_1, \dots, G_m \in \mathrm{r}\mathcal{C}$ such that every *connected* rooted graph from r$\mathcal{C}$ is equivalent under $\equiv_k^{\mathrm{rMSO}}$ to one of $G_1, \dots, G_m$.

The graph M_k is now constructed by taking a copies of G_i for each $i = 1, \dots, m$, and identifying their roots, where $a = a(k)$ is as provided by Lemma 3.3. Let us remark that $\mathcal{G}$ being addable and minor-closed implies $M_k \in \mathrm{r}\mathcal{G}$.

Let $G \in \mathcal{G}$ be an (unrooted) connected graph that contains at least one pendant copy of M_k. Let us fix one such pendant copy (for notational convenience we just identify this copy with M_k from now on). It is convenient to root G at the root r of M_k. Certainly, if Duplicator wins $\mathrm{EHR}_k^{\mathrm{rMSO}}(G, M_k)$, then he also wins the unrooted game $\mathrm{EHR}_k^{\mathrm{MSO}}(G, M_k)$. We consider the rooted version of the game in the remainder of the proof.

Let $G_1, \dots, G_m$ denote the rooted graphs used in the construction of M_k, and let $G_1^1, \dots, G_1^a, \dots, G_m^1, \dots, G_m^a$ be the copies of G_1 until G_m whose roots were identified to create M_k. Let G' denote the (rooted, connected) subgraph of G induced by $(V(G) \setminus V(M_k)) \cup \{r\}$. That is, to obtain G' we remove all vertices of M_k from G except the root. Observe that one can view G as consisting of $G', G_1^1, \dots, G_m^a$, identified along their roots. Also note that $G' \in \mathcal{G}$ since it is a minor of G. Hence, by choice of $G_1, \dots, G_m$, there exists $1 \leq i \leq m$ such that $G' \equiv_k^{\mathrm{rMSO}} G_i$. Without loss of generality $i = 1$. It follows from Lemma 3.2 that $G \equiv_k^{\mathrm{rMSO}} H$, where H is obtained by taking $a+1$ copies of G_1 and a copies of each of $G_2, \dots, G_n$ and identifying the roots. By Lemma 3.3 together with Lemma 3.2 we also have $H \equiv_k^{\mathrm{rMSO}} M_k$. It follows that $G \equiv_k^{\mathrm{rMSO}} M_k$, as required. ■

With Theorem 3.1 in hand, we are now ready to prove the MSO-zero-one law for the random connected graph from an addable, minor-closed class.

Proof of Theorem 1.1: Let $\varphi \in \mathrm{MSO}$ be arbitrary, let k be its quantifier depth and let M_k be as provided by Theorem 3.1. By Corollary 2.15, w.h.p., C_n has a pendant copy of M_k. Thus, by Theorem 3.1, w.h.p., $C_n \equiv_k^{\mathrm{MSO}} M_k$. In particular, this implies that if $M_k \models \varphi$ then $\lim_{n\to\infty} \mathbb{P}(C_n \models \varphi) = 1$ and if, on the other hand, $M_k \models \neg\varphi$ then $\lim_{n\to\infty} \mathbb{P}(C_n \models \varphi) = 0$. ■

3.2 The MSO-convergence law for addable classes

In this section we prove the following more explicit version of Theorem 1.2 above.

Theorem 3.4 *Let $\mathcal{G}$ be an addable, minor-closed class of graphs, let $G_n \in_u \mathcal{G}_n$ and let R be the Boltzmann–Poisson random graph corresponding to $\mathcal{G}$. For every $\varphi \in$ MSO there exists a set $\mathcal{F} = \mathcal{F}(\varphi) \subseteq \mathcal{UG}$ such that*

$$\lim_{n\to\infty} \mathbb{P}(G_n \models \varphi) = \mathbb{P}(R \in \mathcal{F}).$$

In the proof we make use of the following (nearly) trivial consequence of Theorem 2.14.

Corollary 3.5 *In the situation of Theorem 3.4: if $H \in \mathcal{G}$ is any fixed, connected (rooted) graph, then w.h.p.* $\mathrm{Big}(G_n)$ *contains a pendant copy of H.*

Proof: By Theorem 2.14, there exists $\alpha > 0$ such that G_n contains at least αn pendant copies of H w.h.p. Let A_n denote the event that $\mathrm{Big}(G_n)$ contains at least $\alpha n/2$ pendant copies of H, and let B_n denote the event that $\mathrm{Frag}(G_n)$ contains at least $\alpha n/2$ copies of H. We must have $\mathbb{P}(A_n) + \mathbb{P}(B_n) \geq \mathbb{P}(A_n \cup B_n) = 1 - o(1)$. Now observe that, for every fixed $K > 0$ we have

$$\mathbb{P}(B_n) \leq \mathbb{P}(v(\mathrm{Frag}(G_n)) > K) = \mathbb{P}(v(R) > K) + o(1),$$

using Theorem 2.12 (where the first inequality holds for n sufficiently large). The probability $\mathbb{P}(v(R) > K)$ can be made arbitrarily small by choosing K large enough. From this it follows that $\mathbb{P}(B_n) = o(1)$. Hence $\mathbb{P}(A_n) = 1 - o(1)$. ■

Proof of Theorem 3.4: Let $\varphi \in$ MSO be arbitrary, let k be its quantifier depth and let M_k be as provided by Theorem 3.1. By Corollary 3.5, w.h.p., $\mathrm{Big}(G_n)$ contains a pendant copy of M_k. Hence, by Theorem 3.1, we have $\mathrm{Big}(G_n) \equiv_k^{\mathrm{MSO}} M_k$ (w.h.p.). From this it also follows, using Lemma 2.6, that

$$G_n \equiv_k^{\mathrm{MSO}} M_k \cup \mathrm{Frag}(G_n) \quad \text{w.h.p.}$$

(where $\cup$ denotes vertex-disjoint union). In particular we have $\mathbb{P}(G_n \models \varphi) = \mathbb{P}(M_k \cup \mathrm{Frag}(G_n) \models \varphi) + o(1)$. Now let $\mathcal{F} \subseteq \mathcal{UG}$ be the set of all unlabelled graphs $H \in \mathcal{UG}$ such that $M_k \cup H \models \varphi$. It follows using Theorem 2.12 that:

$$\lim_{n\to\infty} \mathbb{P}(G_n \models \varphi) = \lim_{n\to\infty} \mathbb{P}(M_k \cup \mathrm{Frag}(G_n) \models \varphi) = \lim_{n\to\infty} \mathbb{P}(F_n \in \mathcal{F}) = \mathbb{P}(R \in \mathcal{F}),$$

where F_n denotes the isomorphism class (i.e., unlabelled version) of $\mathrm{Frag}(G_n)$. ■

3.3 The FO-zero-one law for surfaces

The proof of the FO-zero-one law for surfaces mimics that of the MSO-zero-one law for the addable case. The main ingredient is the following analogue of Theorem 3.1.

Lemma 3.6 *For every $k \in \mathbb{N}$ there exists $\ell = \ell(k)$ and a planar graph L_k such that following holds. For every connected graph G such that*

- **(i)** *the subgraph of G induced by the vertices at distance at most ℓ from v is planar for every $v \in V(G)$,*
- **(ii)** *G contains a pendant copy of L_k,*

it holds that $G \equiv_k^{\mathrm{FO}} L_k$.

Proof: Recall that, up to logical equivalence, there are only finitely many FO-sentences of quantifier depth $\leq k$. For each such sentence we fix an arbitrary local sentence that is logically equivalent to it (such a local sentence exists by Gaifman's theorem). Let $\mathcal{L}$ be the set of local sentences thus obtained and let $\mathcal{B} = \{\varphi_1, \dots, \varphi_m\}$ be the set of all basic local sequences that appear in the boolean combinations $\mathcal{L}$. Let us set $k' := \max(\mathrm{qd}(\varphi_1), \dots, \mathrm{qd}(\varphi_m))$ equal to the maximum quantifier depth over all these local sentences. For each $1 \leq i \leq m$, we can write

$$\varphi_i = \exists x_1, \dots, x_{n_i} : \left(\bigwedge_{1 \leq a \leq n_i} \psi^{B(x_a, \ell_i)}(x_a) \right) \wedge \left(\bigwedge_{1 \leq a < b \leq n} \mathrm{dist}(x_a, x_b) > 2\ell_i \right).$$

We now set $\ell = \ell(k) = \max(\ell_1, \dots, \ell_m)$, and we let L_k be a path of length $1000 \cdot \max(n_1, \dots, n_m) \cdot \ell$ with a pendant copy of $M_{k'}$ attached to one of its endpoints, where $M_{k'}$ is as provided by Theorem 3.1 for the class of planar graphs. We root L_k at the middle vertex of the long path.

Let us now fix an arbitrary $1 \leq i \leq m$. Let us first suppose that the sentence $\exists x, y : \psi_i^{B(x,\ell_i)}(x) \wedge (\mathrm{dist}(x, y) = \ell_i)$ is not satisfied by any connected planar graph. Since L_k is planar and every point is at distance exactly ℓ_i from some other point, the sentence $\exists x : \psi_i^{B(x,\ell_i)}(x)$ does not hold for L_k. Hence φ_i cannot hold for L_k either. Let G be any graph with the properties **(i)** and **(ii)** listed in the statement of the lemma. Observe that, since G contains a copy of L_k, for every $x \in V(G)$ the subgraph $B(x, \ell_i)$ is connected, planar and x is at distance exactly ℓ_i from some other point. But this shows $\exists x : \psi_i^{B(x_i,\ell_i)}(x)$ cannot hold for G, and hence φ_i does not hold for G either.

Next, let us suppose that the sentence $\exists x, y : \psi_i^{B(x,\ell_i)}(x) \wedge (\mathrm{dist}(x, y) = \ell_i)$ is satisfied by at least one connected planar graph H. We construct a graph G' as follows. We take n_i copies of H and join them to an extra point u, via edges to their y-vertices. We now attach a pendant copy of L_k to u. (See Figure 1 for a depiction.)

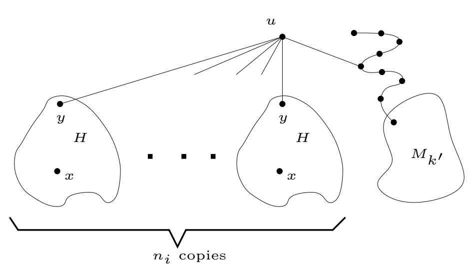

Figure 1: The construction of G'.

Observe that, by construction, $G' \models \varphi_i$. Since G' and L_k are planar, by Theorem 3.1 we have

$$G' \equiv_{k'}^{\mathrm{MSO}} M_{k'} \equiv_{k'}^{\mathrm{MSO}} L_k.$$

So $L_k \models \varphi_i$ as well. Now observe that, because of the special form of φ_i as a basic local sentence (and because L_k will be hanging from the middle vertex of the long path and the ℓ_i-neighbourhoods of vertices on the path but not near the ends are all isomorphic), any graph that contains a pendant copy of L_k will satisfy φ_i as well.

This proves that if G is any graph satisfying the assumptions of the lemma, then $G \models \varphi_i$ if and only if $L_k \models \varphi_i$. Since $1 \leq i \leq m$ was arbitrary, and every FO sentence of quantifier depth at most k can be written as a Boolean combination of $\varphi_1, \dots, \varphi_m$, it follows that $G \equiv_k^{\mathrm{FO}} L_k$ for every G that satisfies the assumptions of the lemma. ■

Proof of Theorem 1.3: The proof closely follows the structure of the proof of Theorem 1.1. Let $\varphi \in \mathrm{FO}$ be arbitrary, let k be its quantifier depth and let ℓ, L_k be as provided by Lemma 3.6. By Corollary 2.20, w.h.p., every ℓ-neighbourhood of every vertex of C_n is planar. By Corollary 2.18, w.h.p., C_n has a pendant copy of L_k. It thus follows from Lemma 3.6 that, w.h.p., $C_n \equiv_k^{\mathrm{FO}} L_k$.

This implies that if $L_k \models \varphi$ then $\lim_{n\to\infty} \mathbb{P}(C_n \models \varphi) = 1$. And if, on the other hand, $L_k \models \neg\varphi$ then $\lim_{n\to\infty} \mathbb{P}(C_n \models \varphi) = 0$.

Since the graph L_k provided by Lemma 3.6 does not depend on the choice of the surface, the same is true for the value of the limiting probability. ■

3.4 The FO-convergence law for surfaces

Theorem 3.7 *Fix a surface S and let $\mathcal{G}$ be the class of all graphs embeddable on S and let $G_n \in_u \mathcal{G}_n$. Let R be the Boltzmann–Poisson random graph corresponding to the class of* planar *graphs $\mathcal{P}$. For every $\varphi \in \mathrm{FO}$ there exists a set $\mathcal{F} = \mathcal{F}(\varphi) \subseteq \mathcal{UP}$ such that*

$$\lim_{n\to\infty} \mathbb{P}(G_n \models \varphi) = \mathbb{P}(R \in \mathcal{F}).$$

Moreover, $\mathcal{F}(\varphi) \subseteq \mathcal{UP}$ does not depend on the surface S.

The proof closely follows that of Theorem 3.4. We again separate out a (nearly) trivial consequence of in this case Theorem 2.17. The proof is completely analogous to that of Corollary 3.5 and is left to the reader.

Corollary 3.8 *If H is any fixed, connected, planar (rooted) graph, then w.h.p.* $\mathrm{Big}(G_n)$ *contains a pendant copy of H.* ■

Proof of Theorem 3.7: Let $\varphi \in \mathrm{FO}$ be arbitrary, let k be its quantifier depth and let ℓ, L_k be as provided by Lemma 3.6. By Theorem 2.19, w.h.p., the ℓ-neighbourhood of every vertex of G_n is planar. By Corollary 3.8, w.h.p., $\mathrm{Big}(G_n)$ has a pendant copy of L_k. It follows by Lemma 3.6 that $\mathrm{Big}(G_n) \equiv_k^{\mathrm{FO}} L_k$ (w.h.p.) and hence also, using Lemma 2.6:

$$G_n \equiv_k^{\mathrm{FO}} L_k \cup \mathrm{Frag}(G_n) \quad \text{w.h.p.}$$

(Here $\cup$ again denotes vertex-disjoint union). Let $\mathcal{F} \subseteq \mathcal{UP}$ denote the set of all unlabelled *planar* graphs H such that $L_k \cup H \models \varphi$, and let $\mathcal{F}'$ denote the set of all unlabelled graphs (not-necessarily planar) with the same property. Using Theorem 2.16, we find that:

$$\lim_{n\to\infty} \mathbb{P}(G_n \models \varphi) = \lim_{n\to\infty} \mathbb{P}(L_k \cup \mathrm{Frag}(G_n) \models \varphi) = \lim_{n\to\infty} \mathbb{P}(F_n \in \mathcal{F}') = \mathbb{P}(R \in \mathcal{F}') = \mathbb{P}(R \in \mathcal{F}),$$

where F_n is the isomorphism class of $\mathrm{Frag}(G_n)$, and R is the Boltzmann–Poisson random graph associated with planar graphs. (The last equality holds because the distribution of R assigns probability zero to non-planar graphs.) It is clear that $\mathcal{F}$ does not depend on the choice of surface, since L_k does not depend on the surface either. ■

4 The limiting probabilities

Throughout this section $\mathcal{G}$ will be an arbitrary addable, minor-closed class. For notational convenience we shall write

$$L_{\mathrm{FO}} := \left\{ \lim_{n\to\infty} \mathbb{P}(G_n \models \varphi) : \varphi \in \mathrm{FO} \right\}, \qquad L_{\mathrm{MSO}} := \left\{ \lim_{n\to\infty} \mathbb{P}(G_n \models \varphi) : \varphi \in \mathrm{MSO} \right\},$$

where $G_n \in_u \mathcal{G}_n$ as usual.

4.1 There is always a gap in the middle

In this section we prove the following lemma about the structure of the logical limit sets of a general addable minor-closed class, which together with Theorem 2.13 proves Proposition 1.5. Notice that $1/G(\rho) \geq 1/\sqrt{e} > 1/2$.

Lemma 4.1 *Let $\mathcal{G}$ be an addable, minor-closed class of graphs. Then* $L_{\mathrm{MSO}} \cap \left(1 - \frac{1}{G(\rho)}, \frac{1}{G(\rho)}\right) = \emptyset$.

Proof: Let $\varphi \in \mathrm{MSO}$ be arbitrary, and let E denote the event that G_n is connected. Observe that if we condition on E, then G_n is distributed like C_n (G_n is then chosen uniformly at random from all connected graphs on n vertices). Since the MSO-zero-one law holds for C_n by Theorem 1.1 we have either $\mathbb{P}(C_n \models \varphi) = 1 - o(1)$ or $\mathbb{P}(C_n \models \varphi) = o(1)$. Let us first assume the former is the case. Then

$$\begin{array}{rcl}
\lim_{n\to\infty} \mathbb{P}(G_n \models \varphi) & = & \lim_{n\to\infty} \mathbb{P}(G_n \models \varphi | E) \cdot \mathbb{P}(E) + \mathbb{P}(G_n \models \varphi | E^c) \cdot \mathbb{P}(E^c) \\
& \geq & \lim_{n\to\infty} \mathbb{P}(G_n \models \varphi | E) \cdot \mathbb{P}(E) \\
& = & \lim_{n\to\infty} \mathbb{P}(C_n \models \varphi) \cdot \mathbb{P}(E) \\
& = & 1 \cdot 1/G(\rho),
\end{array}$$

where for the last equality we used the general limit in (7) above. Now suppose $\mathbb{P}(C_n \models \varphi) = o(1)$. Then $\lim_{n\to\infty} \mathbb{P}(G_n \models \neg\varphi) \geq 1/G(\rho)$ by the above, hence $\lim_{n\to\infty} \mathbb{P}(G_n \models \varphi) \leq 1 - 1/G(\rho)$. ■

4.2 The closure is a finite union of intervals

Here we prove the following more detailed version of Theorem 1.6. Note that both L_{MSO} and L_{FO} are countable sets since the set of MSO-sentences is countable.

Theorem 4.2 *Let $\mathcal{G}$ be an addable, minor-closed class of graphs and let R be the corresponding Boltzmann–Poisson random graph. Then*

$$\mathrm{cl}(L_{\mathrm{MSO}}) = \mathrm{cl}(L_{\mathrm{FO}}) = \{\mathbb{P}(R \in \mathcal{F}) : \mathcal{F} \subseteq \mathcal{UG}\},$$

is a finite union of closed intervals.

Before starting the proof of Theorem 4.2, we will derive a number of auxiliary lemmas. From Theorem 3.4 we see immediately that $L_{\mathrm{FO}} \subseteq L_{\mathrm{MSO}} \subseteq \{\mathbb{P}(R \in \mathcal{F}) : \mathcal{F} \subseteq \mathcal{UG}\}$. The next lemma shows that L_{FO} is in fact dense in $\{\mathbb{P}(R \in \mathcal{F}) : \mathcal{F} \subseteq \mathcal{UG}\}$.

Lemma 4.3 *For every $\mathcal{F} \subseteq \mathcal{UG}$ and every $\varepsilon > 0$ there is a $\varphi \in \mathrm{FO}$ such that*

$$|\mathbb{P}(R \in \mathcal{F}) - \lim_{n\to\infty} \mathbb{P}(G_n \models \varphi)| \leq \varepsilon.$$

Proof: First note that it in fact suffices to consider only *finite* $\mathcal{F} \subseteq \mathcal{UG}$. (To see this, notice that there is always a finite $\mathcal{F}' \subseteq \mathcal{F}$ such that $\mathbb{P}(R \in \mathcal{F}') \geq \mathbb{P}(R \in \mathcal{F}) - \varepsilon/2$). Let us thus assume $\mathcal{F}$ is finite.

Let us pick a K such that $\mathbb{P}(v(R) > K) < \varepsilon$ and let $\mathrm{Frag}_K(G)$ denote the union of all components of G of order at most K. Let us observe that, for every $F \in \mathcal{F}$, the event $\{\mathrm{Frag}_K(G_n) \cong F\}$ is FO-expressible. (We simply stipulate, for each of the connected graphs $H \in \mathcal{UC}$ on at most K vertices, how many components isomorphic to H the random graph G_n should contain.) Since $\mathcal{F}$ is finite, the event $\{\mathrm{Frag}_K(G_n) \in \mathcal{F}\} = \bigcup_{F\in\mathcal{F}} \{\mathrm{Frag}_K(G_n) \cong F\}$ is therefore also FO-expressible. Observe that

$$\begin{array}{rcl}
\lim\limits_{n\to\infty} \mathbb{P}\Big[\mathrm{Frag}_K(G_n) \in \mathcal{F}\Big] & \leq & \lim\limits_{n\to\infty} \mathbb{P}\Big[\mathrm{Frag}(G_n) \in \mathcal{F} \text{ or } v(\mathrm{Frag}(G_n)) > K\Big] \\
& \leq & \lim\limits_{n\to\infty} \mathbb{P}\Big[\mathrm{Frag}(G_n) \in \mathcal{F}\Big] + \lim\limits_{n\to\infty} \mathbb{P}\Big[v(\mathrm{Frag}(G_n)) > K\Big] \\
& < & \mathbb{P}(R \in \mathcal{F}) + \varepsilon.
\end{array}$$

Similarly,

$$\begin{array}{rcl}\lim\limits_{n\to\infty} \mathbb{P}\Big[\mathrm{Frag}_K(G_n) \in \mathcal{F}\Big] & \geq & \lim\limits_{n\to\infty} \mathbb{P}\Big[\mathrm{Frag}(G_n) \in \mathcal{F} \text{ and } v(\mathrm{Frag}(G_n)) \leq K\Big] \\ & \geq & \lim\limits_{n\to\infty} \mathbb{P}\Big[\mathrm{Frag}(G_n) \in \mathcal{F}\Big] - \lim\limits_{n\to\infty} \mathbb{P}\Big[v(\mathrm{Frag}(G_n)) > K\Big] \\ & > & \mathbb{P}(R \in \mathcal{F}) - \varepsilon. \end{array}$$

This concludes the proof of the lemma. ■

Having established that L_{FO} is a dense subset of $\{\mathbb{P}(R \in \mathcal{F}) : \mathcal{F} \subseteq \mathcal{UG}\}$, to prove Theorem 4.2 it suffices to show that this last set is a finite union of intervals. For the remainder of this section the random graph G_n will no longer play any role, and all mention of probabilities, events etc. are with respect to the Boltzmann–Poisson random graph R.

Let us order the unlabelled graphs $G_1, G_2, \dots \in \mathcal{UG}$ in such a way that the probabilities $p_i := \mathbb{P}(R = G_i)$ are non-increasing. By Corollary 2.3, to prove Theorem 4.2 it suffices to show that $p_i \leq \sum_{j>i} p_j$ for all sufficiently large i. For $k \in \mathbb{N}$, let us write:

$$E_k := \Big\{R \text{ contains no component with} < k \text{ vertices and exactly one component with } k \text{ vertices}\Big\},$$
$$q_k := \mathbb{P}(E_k).$$

Lemma 4.4 *For every $k \in \mathbb{N}$, there is a set $A_k \subseteq \mathbb{N}$ such that $q_k = \sum_{i \in A_k} p_i$. Moreover, the sets A_k are disjoint.*

Proof: Phrased differently, the lemma asks for a $\mathcal{F} \subseteq \mathcal{UG}$ such that we can write $\mathbb{P}(E_k) = \sum_{H \in \mathcal{F}} \mathbb{P}(R = H)$. But this is obvious. That the sets A_k are disjoint follows immediately from the fact that the events E_k are disjoint. ■

For each $k \in \mathbb{N}$, let Z_k denote the number of components of R of order k and let us write

$$\mu_k := \sum_{H \in \mathcal{UC}_k} \frac{\rho^k}{\mathrm{aut}(H)}.$$

(Recall that $\mathcal{UC}_k$ denotes the set of unlabelled, connected graphs from $\mathcal{G}$ on exactly k vertices.) Since the sum of independent Poisson random variables is again Poisson-distributed, it follows from Lemma 2.11 that $Z_1, \dots, Z_k$ are independent Poisson random variables with means $\mathbb{E}Z_i = \mu_i$. Hence we have:

$$q_k = \mathbb{P}(\mathrm{Po}(\mu_1) = 0) \cdots \mathbb{P}(\mathrm{Po}(\mu_{k-1}) = 0)\mathbb{P}(\mathrm{Po}(\mu_k) = 1) = \mu_k \mathrm{e}^{-(\mu_1 + \cdots + \mu_k)}. \quad (8)$$

Lemma 4.5 *We have $\lim\limits_{k\to\infty} q_k = 0$ and $\lim\limits_{k\to\infty} \frac{q_{k+1}}{q_k} = 1$.*

Proof: For $H \in \mathcal{UC}_k$ the quantity $k!/\mathrm{aut}(H)$ is exactly the number of labelled graphs $G \in \mathcal{C}_k$ that are isomorphic to H. It follows that

$$\mu_k = \sum_{H \in \mathcal{UC}_k} \frac{\rho^k}{\mathrm{aut}(H)} = \sum_{G \in \mathcal{C}_k} \frac{\rho^k}{k!} = \frac{|\mathcal{C}_k|}{k!} \cdot \rho^k.$$

We thus have that

$$\sum_{k=1}^{\infty} \mu_k = C(\rho) \leq G(\rho) < \infty,$$

(where $C(z)$ resp. $G(z)$ denotes the exponential generating function of $\mathcal{C}$ resp. $\mathcal{G}$). In particular we have $\mu_k \to 0$ as $k \to \infty$. This immediately also gives that $q_k \to 0$ as $k \to \infty$. Now recall that, according to Theorem 2.9, we have $\frac{(k+1)|\mathcal{C}_k|}{|\mathcal{C}_{k+1}|} \to \rho$ as $k \to \infty$. We therefore have:

$$\lim_{k\to\infty} \frac{q_{k+1}}{q_k} = \lim_{k\to\infty} \frac{\mu_{k+1}}{\mu_k} \cdot \mathrm{e}^{-\mu_{k+1}} = \lim_{k\to\infty} \frac{\rho|\mathcal{C}_{k+1}|}{(k+1)|\mathcal{C}_k|} \cdot \mathrm{e}^{-\mu_{k+1}} = 1,$$

as required. ■

We are now ready to complete the proof of Theorem 4.2.

Proof of Theorem 4.2: As observed previously, it suffices to show that there exists some $i_0 \in \mathbb{N}$ such that $p_i \le \sum_{j>i} p_j$ for all $i \ge i_0$. By Lemma 4.5, there is an index k_0 such that $q_{k+1} \ge 0.9 \cdot q_k$ for all $k \ge k_0$. We now fix an index i_0 with the property that $p_{i_0} < q_{k_0}$.

Let $i \ge i_0$ be arbitrary and let $k \ge k_0$ be the largest index such that $q_k \ge p_i$. (Such a k exists since $0 < p_i \le p_{i_0} < q_{k_0}$ and $q_k \to 0$.) By choice of k we must have $p_i > q_{k+\ell}$ for all $\ell \ge 1$. Since $k \ge k_0$ we have that

$$q_{k+1} + q_{k+2} + \cdots \ge (0.9 + (0.9)^2 + \dots)p_i = 9p_i > p_i$$

Recall that, by Lemma 4.4, there are disjoint sets $A_m \subseteq \mathbb{N}$ such that $q_m = \sum_{j \in A_m} p_j$ for all $m \in \mathbb{N}$. Because $p_i > q_{k+\ell}$ for all $\ell \ge 1$ and $(p_n)_n$ is non-increasing, we must have that $i < j$ for all $j \in A := \bigcup_{m>k} A_m$. It follows that

$$p_i < \sum_{\ell > k} q_\ell = \sum_{j \in A} p_j \le \sum_{j>i} p_j,$$

as required. Since $i \ge i_0$ was arbitrary, this concludes the proof of Theorem 4.2. ■

4.3 Obtaining the closure explicitly for forests

Here we prove Theorem 1.7 above. For the class of forests $\mathcal{F}$, it is well known that $\rho = \mathrm{e}^{-1}$ and $G(\rho) = \mathrm{e}^{1/2}$, see for example [15, Theorem IV.8] for a framework in which one can construct such explicit constants, and [5, p. 470, Section 2] for an explanation of that particular value.

Our plan for the proof of Theorem 1.7 is of course to apply Corollary 2.3. We wish to find an ordering $F_1, F_2, \dots$ of all unlabelled forests $\mathcal{UF}$ with the property that the probabilities $p_i := \mathbb{P}(R = F_i)$ are non-increasing (here, R is the Boltzmann–Poisson random graph corresponding to $\mathcal{F}$), and to to determine exactly for which values i the inequality $p_i > \sum_{j>i} p_j$ holds.

To this end, we first 'guess' the initial part of the order. Let the graphs $F_1, \dots, F_9$ be as defined in Figure 2.

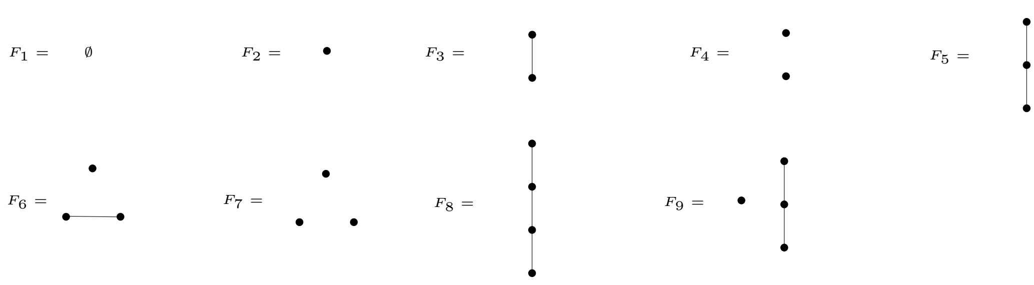

Figure 2: The forests $F_1, \dots, F_9$.

By substituting $\rho = \mathrm{e}^{-1}, G(\rho) = \mathrm{e}^{1/2}$ in (6), we find that the probabilities corresponding to $F_1, \dots, F_9$ are

$$\begin{aligned} &p_1 = \mathrm{e}^{-1/2}, && p_2 = \mathrm{e}^{-3/2}, && p_3 = p_4 = \mathrm{e}^{-5/2}/2, \\ &p_5 = p_6 = \mathrm{e}^{-7/2}/2, && p_7 = \mathrm{e}^{-7/2}/6, && p_8 = p_9 = \mathrm{e}^{-9/2}/2. \end{aligned} \tag{9}$$

It is readily seen that $p_1 \ge \cdots \ge p_9$. Let us remark that, as the reader can easily check, every forest F that is not isomorphic to one of $F_1, \dots, F_9$ has either five or more vertices or it has exactly four vertices and $\mathrm{aut}(F) \ge 4$. (In the second case it is either $K_{1,3}$, or four isolated vertices, or two vertex-disjoint edges, or an edge plus two isolated vertices.) Hence, if F is not isomorphic to one of $F_1, \ldots, F_9$ then $P(R = F) \le \max(\mathrm{e}^{-9/2}/4, \mathrm{e}^{-11/2}) = \mathrm{e}^{-11/2} < \mathrm{e}^{-9/2}/2 = p_9$. This shows that we guessed correctly, and $F_1, \ldots, F_9$ are indeed the first nine forests in our order.

Lemma 4.6 $\displaystyle\sum_{F \in \mathcal{UF}_n} \frac{1}{\mathrm{aut}(F)} > e$ *for every* $n \ge 6$.

Proof: Let $n \ge 6$. We will explicitly describe enough unlabelled forests on n vertices with small automorphism groups to make the sum exceed e. Figure 3 shows them from left to right in the special case $n = 6$.

For every $n \ge 6$, the following five forests all have at most two automorphisms: a path on n vertices, the union of a path on $n-1$ vertices and an isolated vertex, a path on $n-1$ vertices with a leaf attached to its second vertex, and a path on $n-1$ vertices with a leaf attached to its third vertex, the union of an isolated vertex and a path on $n-2$ vertices with a leaf attached to its second vertex. The following forests both have exactly four automorphisms for every $n \ge 6$: the union of a path on $n-2$ vertices with two isolated vertices, the union of a path on $n-2$ vertices with a path on two vertices.

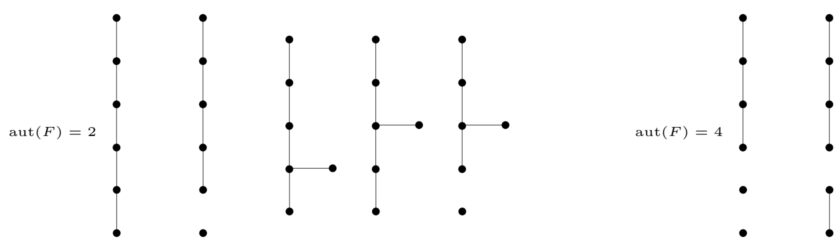

Figure 3: Some forests with small automorphism groups.

For every $n \ge 6$, the seven forests just described are pairwise non-isomorphic. We thus have $\sum_{F \in \mathcal{UF}_n} \frac{1}{\mathrm{aut}(F)} \ge 5 \cdot (1/2) + 2 \cdot (1/4) = 3 > e$. This proves the lemma. ■

Lemma 4.7 *The only indices k for which the inequality $p_k > \sum_{j>k} p_j$ is satisfied are $k = 1, 2$.*

Proof: Since $\sum_{j>k} p_j = 1 - (p_1 + \cdots + p_k)$ we have that $p_k > \sum_{j>k} p_j$ if and only if $p_1 + \cdots + p_{k-1} + 2p_k > 1$. The reader can easily check using the expressions given in (9) that $k = 1, 2$ are the only values of $k \le 9$ for which this inequality holds.

Let $k \ge 10$ be arbitrary, and recall that in this case, as remarked previously, we have $p_k \le \mathrm{e}^{-11/2}$. Let $n \ge 6$ be the unique integer such that

$$\mathrm{e}^{-(n+1/2)} < p_k \le \mathrm{e}^{-(n-1/2)}.$$

Then $\mathbb{P}(R = F) = \mathrm{e}^{-(n+1/2)}/\,\mathrm{aut}(F) < p_k$, for every $F \in \mathcal{UF}_n$. In other words, every forest on n vertices must come *after* position k in our ordering of the unlabelled forests. Using Lemma 4.6 it now follows that:

$$\sum_{j>k} p_j \ge \sum_{F \in \mathcal{UF}_n} \mathbb{P}(R = F) = \mathrm{e}^{-(n+1/2)} \cdot \sum_{F \in \mathcal{UF}_n} \frac{1}{\mathrm{aut}(F)} > \mathrm{e}^{-(n-1/2)} \ge p_k.$$

So the inequality $p_k > \sum_{j>k} p_j$ indeed fails for all $k \ge 10$. This proves the lemma. ■

Proof of Theorem 1.7: By Lemma 4.7 and Corollary 2.3, we see that

$$\mathrm{cl}(L_{\mathrm{MSO}}) = \bigcup_{a,b \in \{0,1\}} \Big[ap_1 + bp_2, ap_1 + bp_2 + (1 - p_1 - p_2)\Big].$$

Filling in the values for p_1, p_2 from (9), we see that we get exactly the four intervals shown in the statement of the theorem. ■

4.4 Obtaining the closure explicitly for planar graphs

Here we prove the following more detailed version of Theorem 1.8 above:

Theorem 4.8 *If $\mathcal{G} = \mathcal{P}$ is the class of planar graphs, ρ the radius of convergence of its exponential generating function G, and if*

$$\lambda_{a,b,c,d,e} := \frac{a + b\rho + \frac{c}{2}\rho^2 + (\frac{d}{2} + \frac{e}{6})\rho^3}{G(\rho)} \quad , \qquad \ell := 1 - \frac{1 + \rho + \rho^2 + \frac{4}{3}\rho^3}{G(\rho)} \quad , \tag{10}$$

then

$$\mathrm{cl}(L_{\mathrm{MSO}}) = \bigcup_{\substack{a,b \in \{0,1\},\\ c,d,e \in \{0,1,2\}}} [\lambda_{a,b,c,d,e}, \lambda_{a,b,c,d,e} + \ell] \,.$$

In particular, $\mathrm{cl}(L_{\mathrm{MSO}})$ *is the union of 108 disjoint intervals each of length* $1 - \frac{1}{G(\rho)}(1+\rho+\rho^2+\frac{4}{3}\rho^3)$, *a number close to* $5 \cdot 10^{-6}$.

The proof closely follows the structure of the proof from the previous section. Our plan is again to find (the initial part of) an ordering $G_1, G_2, \dots$ of $\mathcal{UP}$ such that the sequence of probabilities $p_k := \mathbb{P}(R = G_k)$ is non-increasing, and to determine precisely for which indices k the condition $p_k > \sum_{j>k} p_j$ holds. Again we start by 'guessing' the first few graphs in the ordering. Let the graphs $G_1, \dots, G_{19}$ be as defined in Figure 4.

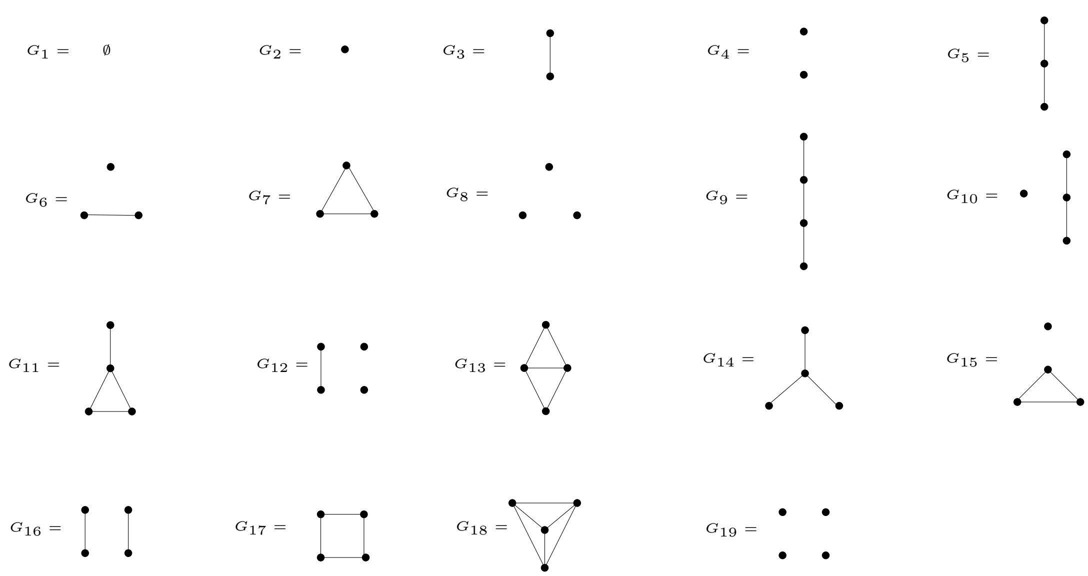

Figure 4: The graphs $G_1, \dots, G_{19}$.

(Observe that these are precisely all unlabelled graphs on at most four vertices – including the empty graph G_1.) The corresponding probabilities are

$$\begin{aligned}
&p_1 = \tfrac{1}{G(\rho)}, && p_2 = \tfrac{\rho}{G(\rho)}, && p_3 = p_4 = \tfrac{\rho^2}{2G(\rho)}, && p_5 = p_6 = \tfrac{\rho^3}{2G(\rho)},\\
&p_7 = p_8 = \tfrac{\rho^3}{6G(\rho)}, && p_9 = p_{10} = p_{11} = \tfrac{\rho^4}{2G(\rho)}, && p_{12} = p_{13} = \tfrac{\rho^4}{4G(\rho)}, && p_{14} = p_{15} = \tfrac{\rho^4}{6G(\rho)},\\
&p_{16} = p_{17} = \tfrac{\rho^4}{8G(\rho)}, && p_{18} = p_{19} = \tfrac{\rho^4}{24G(\rho)}.
\end{aligned} \tag{11}$$

To decide for which indices the tail-exceeds-term condition $p_i \le \sum_{j>i} p_j$ holds (and to check that $p_1, \dots, p_{19}$ are in non-increasing order and that all graphs on at least five vertices satisfy $p_i \le p_{19}$), we need more detailed information on the values of ρ and $G(\rho)$ for planar graphs. Such information is available form the work of Giménez and the third author [17], who determined both quantities precisely as the solution of a (non-polynomial) system of equations. This system in particular enables us to compute the numbers ρ and $G(\rho)$ to any desired degree of accuracy. The following approximations suffices for the present purpose:

Lemma 4.9 *With G the exponential generating function of labelled planar graphs, and ρ its radius of convergence,*

$$0.03672841251 \le \rho \le 0.03672841266, \quad 0.96325282112 \le 1/G(\rho) \le 0.96325282254.$$

Our proof of Theorem 4.8 depends in a delicate way on the numerical values of ρ and $G(\rho)$. For example, one can choose approximations of ρ and $1/G(\rho)$ that agree with Lemma 4.9 in the first five digits, but differ in the sixth, and that will result in a different number of intervals in Theorem 4.8. The approximations in Lemma 4.9 can be computed easily using a computer algebra package. For completeness we provide a proof that can be checked by hand, in a supporting document [21].

Getting back to the current proof, let us first observe that, now that we know $\rho < 1/24$, it is indeed true that $p_1 \ge \cdots \ge p_{19}$ and that for $i > 19$ we have $p_i \le \frac{\rho^5}{G(\rho)} < \frac{\rho^4}{24G(\rho)} = p_{19}$, as G_i must have at least five vertices. Thus $G_1, \ldots, G_{19}$ are indeed the first nineteen unlabelled planar graphs, when the unlabelled planar graphs are ordered by non-increasing value of $\mathbb{P}(R = G_i)$.

The following lower bound will be sufficient for our purposes:

Lemma 4.10 *For $n \ge 6$ it holds that* $\displaystyle\sum_{H \in \mathcal{UP}_n} \frac{1}{\mathrm{aut}(H)} > 30.$

Proof: For notational convenience let us define $\varphi(n) := \sum_{H \in \mathcal{UP}_n} \frac{1}{\mathrm{aut}(H)}$. Because of

$$|\mathcal{P}_n| = \sum_{H \in \mathcal{UP}_n} \frac{n!}{\mathrm{aut}(H)} = n! \cdot \varphi(n),$$

we have $\varphi(n) = |\mathcal{P}_n|/n!$. Let us observe that $n|\mathcal{P}_{n-1}| \le |\mathcal{P}_n|$ for every $n \ge 2$. This is because given an arbitrary element of $\mathcal{P}_{n-1}$, already the possibility to add the vertex n and then either join it to exactly one existing vertex, or leave it isolated, creates $n - 1 + 1 = n$ distinct planar labelled graphs on $[n]$, and all $n|\mathcal{P}_{n-1}|$ elements of $\mathcal{P}_n$ thus created are distinct. From this it follows φ is monotone non-decreasing in n, as $\varphi(n)/\varphi(n-1) = |\mathcal{P}_n|/(n|\mathcal{P}_{n-1}|) \ge 1$. It therefore suffices to prove $\varphi(6) > 30$, or equivalently, $|\mathcal{P}_6| > 21600$. Now we use work of Bodirsky, Kang and Gröpl: the number of all labelled planar graphs on six vertices and with m edges is given, for all possible values $0 \le m \le 12$, in the fifth row of the table in [6, Fig. 1] (the notation $G^{(0)}(n,m)$ is defined on p. 379). The sum of these numbers is $|\mathcal{P}_6|$, and equals 32071, which is strictly greater than 21600. ■

Lemma 4.11 *The only indices k for which the inequality $p_k > \sum_{j>k} p_j$ holds are $k = 1, 2, 4, 6, 8$.*

Proof: Recall that $p_k > \sum_{j>k} p_j$ if and only if $p_1 + \cdots + p_{k-1} + 2p_k > 1$. As it happens, the estimates on ρ and $1/G(\rho)$ provided by Lemma 4.9, together with the expressions (11) suffice to determine for which $k \le 19$ the inequality holds. We leave the routine arithmetic computations verifying this to the reader. To see that the inequality holds for $k = 1, 2, 4, 6, 8$, it suffices to do explicit calculations with the lower bounds, and to see that it fails for $k = 11, 13, 15, 17, 19$, it suffices to do explicit calculations with the upper bounds provided by Lemma 4.9. Observe that $p_k = p_{k+1}$ implies that $p_k \le \sum_{j>k} p_j$, so that the inequality automatically fails for $k = 3, 5, 7, 9, 10, 12, 14, 16, 18$.

To complete the proof of Lemma 4.11, we are now left with $k \ge 20$. Let $k \ge 20$ be arbitrary. Since $G_1, \ldots, G_{19}$ are all the unlabelled graphs on at most four vertices, we must have $v(G_k) \ge 5$, so the formula in Definition 2.10 implies $p_k \leq \rho^5/G(\rho)$. Let $n \geq 6$ be the unique integer such that

$$\frac{\rho^n}{G(\rho)} < p_k \leq \frac{\rho^{n-1}}{G(\rho)}.$$

Then $\mathbb{P}(R = H) = \frac{\rho^n}{\mathrm{aut}(H)G(\rho)} < p_k$ for every $H \in \mathcal{UP}_n$. Hence, every graph on n vertices must come *after* position k in the ordering. By Lemma 4.10 and the bound $\rho > 1/30$ from Lemma 4.9,

$$\sum_{j>k} p_j \geq \sum_{H \in \mathcal{UP}_n} \mathbb{P}(R = H) = \frac{\rho^n}{G(\rho)} \cdot \sum_{H \in \mathcal{UP}_n} \frac{1}{\mathrm{aut}(H)} > \frac{\rho^{n-1}}{G(\rho)} \geq p_k,$$

completing the proof. ∎

Proof of Theorem 4.8: The result follows immediately from Lemma 4.11 via an application of Corollary 2.3. Note that $\sum_{j>8} p_j = 1 - \frac{1}{G(\rho)}(1 + \rho + \rho^2 + \frac{4}{3}\rho^3)$ and that c, d, e in the expression given in the theorem take values in $\{0, 1, 2\}$ because $p_3 = p_4$ and $p_5 = p_6$ and $p_7 = p_8$.

That the $2 \cdot 2 \cdot 3 \cdot 3 \cdot 3 = 108$ intervals thus defined are all disjoint follows from the fact that their left endpoints always differ by at least p_8 while each interval has length $\sum_{j>8} p_j < p_8$. ∎

5 The non-addable case

5.1 The MSO-zero-one law for bounded component size

Here we prove Theorem 1.9. In this subsection, we fix $k \in \mathbb{N}$ and $\mathcal{G}$ will be the class of all graphs whose components have at most k vertices. We need the following lemma on the number of components of $G_n \in_u \mathcal{G}_n$ isomorphic to a given graph.

Lemma 5.1 *Let H be a fixed, connected graph from $\mathcal{G}$, let K be an arbitrary constant and let Z_n denote the number of components of G_n that are isomorphic to H. Then $Z_n(H) > K$ w.h.p.*

Proof: Let us write $r = v(H)$. Using Corollary 2.23, we see that

$$\mathbb{E}Z_n = \frac{\binom{n}{r} \cdot \frac{r!}{\mathrm{aut}(H)} \cdot |\mathcal{G}_{n-r}|}{|\mathcal{G}_n|} \sim \frac{c^r n^{r/k}}{\mathrm{aut}(H)}.$$

Similarly, we have

$$\mathbb{E}Z_n(Z_n - 1) = \frac{\binom{n}{r} \cdot \binom{n-r}{r} \cdot \left(\frac{r!}{\mathrm{aut}(H)}\right)^2 \cdot |\mathcal{G}_{n-2r}|}{|\mathcal{G}_n|} \sim \left(\frac{c^r n^{r/k}}{\mathrm{aut}(H)}\right)^2.$$

It follows that $\mathrm{Var}(Z_n) = \mathbb{E}Z_n^2 - (\mathbb{E}Z_n)^2 = o((\mathbb{E}Z_n)^2)$. We can thus write

$$\mathbb{P}(Z_n \leq K) \leq \mathbb{P}(|Z_n - \mathbb{E}Z_n| \geq \mathbb{E}Z_n/2) \leq 4\,\mathrm{Var}\, Z_n/(\mathbb{E}Z_n)^2 = o(1),$$

where the first inequality holds for n sufficiently large (as $\mathbb{E}Z_n \to \infty$) and we have used Chebychev's inequality for the second inequality. Hence $Z_n > K$ w.h.p., as required. ∎

Proof of Theorem 1.9: Let us fix a $\varphi \in$ MSO and let k be its quantifier depth. Let $a = a(k)$ be as provided by Lemma 2.7, let $H_1, \dots, H_m$ be all unlabelled, connected graphs on at most k vertices and set $H := aH_1 \cup \dots \cup aH_m$. (So H is the vertex disjoint union of a copies of H_i, for every i.) By Lemma 5.1, w.h.p., G_n has at least a components isomorphic to H_i for each i (and no other components). Using Lemmas 2.6 and 2.7 it thus follows that, w.h.p., $G_n \equiv_k^{\mathrm{MSO}} H$. So if $H \models \varphi$ then $\lim_{n\to\infty} \mathbb{P}(G_n \models \varphi) = 1$ and otherwise $\lim_{n\to\infty} \mathbb{P}(G_n \models \varphi) = 0$. ∎

5.2 A MSO-sentence without a limiting probability, for paths

Note that it is possible to ask, in a MSO-sentence, for a proper two-colouring of the graph, such that there are two vertices of degree one with the same colour. If this sentence is true of a path, the path must have odd order. Otherwise it fails. Thus we have:

Corollary 5.2 *There exists a* $\varphi \in \mathrm{MSO}$ *such that* $\mathbb{P}(C_n \models \varphi) = \begin{cases} 1 & \text{if } n \text{ is odd,} \\ 0 & \text{if } n \text{ is even.} \end{cases}$

5.3 The FO-zero-one law fails for caterpillars

Let φ be the FO sentence which formalizes 'there are two distinct vertices that have degree five and exactly one neighbour of degree at least two'. Then clearly a caterpillar satisfies φ if and only if both ends of its spine have degree five. (Here and elsewhere, we define the spine of a caterpillar as the path that consists of all vertices of degree at least two.) The following lemma shows the FO-zero-one law fails for C_n, the random caterpillar.

Proposition 5.3 *If* φ *is as above, then* $\lim_{n\to\infty} \mathbb{P}(C_n \models \varphi) = \left(\frac{\rho^4}{4!(e^\rho - 1)}\right)^2$, *where* ρ *is the unique real root of* $x\mathrm{e}^x = 1$.

Proof: An oriented (labelled) caterpillar is a caterpillar, with ≥ 2 vertices on its spine, on which we choose a "direction" for the spine. In other words, an oriented caterpillar is a sequence of at least two stars, such that the first and the last star have at least two vertices. Let a and b be the endpoints of the spine. We compute the joint distribution of the number of leaves attached to a and b. It follows (cf. [15, Chapter II]) by basic theory of exponential generating functions (EGF) that, with number of vertices as size function, the EGF of stars is $x\mathrm{e}^x$, of stars with at least two vertices is $x\mathrm{e}^x - x$ and of oriented caterpillars is

$$C(x) = \frac{(x\mathrm{e}^x - x)^2}{1 - x\mathrm{e}^x}.$$

The numerator encodes the first and last stars, and the denominator the sequence (possibly empty) of intermediate stars.

Since $C(x)$ is a meromorphic function with a simple pole at ρ, we can apply the results from [15, Section IV.5]). It follows that the number of oriented caterpillars on n vertices satisfies

$$[x^n]C(x) \sim c \cdot \rho^{-n} n!, \tag{12}$$

for some constant $c > 0$.

We introduce variables u and w marking, respectively, the number of leaves attached to a and b. The associated EGF is

$$C(x, u, w) = \frac{x(\mathrm{e}^{ux} - 1)x(\mathrm{e}^{wx} - 1)}{1 - x\mathrm{e}^x}.$$

The probability that $\deg(a) = i + 1$ and $\deg(b) = j + 1$ is given by

$$\frac{[x^n u^i w^j]C(x, u, w)}{[x^n]C(x)} = \frac{1}{i!j!} \frac{[x^n]x^{i+j}(\mathrm{e}^x - 1)^{-2}C(x)}{[x^n]C(x)} \sim \frac{\rho^i}{i!(\mathrm{e}^\rho - 1)} \frac{\rho^j}{j!(\mathrm{e}^\rho - 1)}.$$

This is because $x^{i+j}(e^x - 1)^2$ is analytic, so that $[x^n]x^{i+j}(\mathrm{e}^x - 1)^{-2}C(x) \sim \rho^{i+j}(\mathrm{e}^\rho - 1)^{-2}[x^n]C(x)$.

We see that asymptotically the number of leaves attached to a and b are independent random variables, each distributed like one plus a Poisson-variable that is conditioned to be positive. In other words, if O_n is chosen uniformly at random from all oriented caterpillars on n vertices then:

$$\lim_{n\to\infty} \mathbb{P}(O_n \models \varphi) = \left(\frac{\rho^4}{4!(e^\rho - 1)}\right)^2.$$

Let E_n denote the event that C_n, the random (unoriented) caterpillar, is not a star (i.e. the spine has at least two vertices). Then we have that

$$\mathbb{P}(C_n \models \varphi | E_n) = \mathbb{P}(O_n \models \varphi),$$

since every unoriented, labelled caterpillar with at least two vertices in the spine corresponds to exactly two oriented, labelled caterpillars. Since there are n stars on n vertices, and the total number of caterpillars is at least $n!/2$ (the number of paths), we have $\mathbb{P}(E_n^c) = o(1)$. It follows that

$$\lim_{n\to\infty} \mathbb{P}(C_n \models \varphi) = \lim_{n\to\infty} \mathbb{P}(C_n \models \varphi | E_n)\mathbb{P}(E_n) + \lim_{n\to\infty} \mathbb{P}(C_n \models \varphi | E_n^c)\mathbb{P}(E_n^c) = \left(\frac{\rho^4}{4!(e^\rho - 1)}\right)^2,$$

as claimed. ■

5.4 The proof of parts (iii) of Theorems 1.10 and 1.11

The following general result proves parts **(iii)** of Theorems 1.10 and 1.11 in a single stroke. We recall that for decomposable classes, we have the 'exponential formula' $G(z) = \exp(C(z))$, where $C(z)$ is the exponential generating function for the connected graphs $\mathcal{C} \subseteq \mathcal{G}$. Moreover, both forests of paths and forests of caterpillars are smooth by Theorem 2.22. Both classes are also decomposable and they satisfy $G(\rho) = \infty$ (in view of $C(\rho) = \infty$ [7, Table 1] and $G(z) = \mathrm{e}^{C(z)}$). Let us also note that both C and G have the same radius of convergence.

Theorem 5.4 *Let $\mathcal{G}$ be a decomposable, smooth, minor-closed class satisfying $G(\rho) = \infty$ and let $G_n \in_u \mathcal{G}_n$. Then there is a family of* FO *properties Φ such that their limiting probabilities exist, and*

$$\mathrm{cl}\left(\left\{\lim_{n\to\infty} \mathbb{P}(G_n \models \varphi) : \varphi \in \Phi\right\}\right) = [0, 1].$$

The proof of this theorem will make use of the following corollary to Lemma 2.21.

Corollary 5.5 *With $\mathcal{G}$, G_n and ρ as in Lemma 2.21, $\mathcal{H} \subseteq \mathcal{UG}$ any set of (unlabelled) connected graphs from $\mathcal{G}$, we set $\mu(H) := \rho^{v(H)}/\operatorname{aut}(H)$ for any $H \in \mathcal{H}$ and $\mu(\mathcal{H}) := \sum_{H\in\mathcal{H}} \mu(H)$. If $\mu(\mathcal{H}) = \infty$ then, for any constant $K > 0$, w.h.p. G_n has at least K components isomorphic to members of $\mathcal{H}$.*

Proof: For any finite subset $\mathcal{H}' \subseteq \mathcal{H}$ and any $H \in \mathcal{H}'$ we denote by $N_n(H)$ the number of components of G_n that are isomorphic to H, and by $N_n(\mathcal{H}') = \sum_{H\in\mathcal{H}'} N_n(H)$ the number of components of G_n isomorphic to some member of $\mathcal{H}'$. A sum of independent Poisson random variables being Poisson, it follows from Lemma 2.21 that $N_n(\mathcal{H}') \to_{\mathrm{TV}} Z$, where Z is a Poisson random variable with mean $\mathbb{E}Z = \mu(\mathcal{H}') = \sum_{H\in\mathcal{H}'} \mu(H)$. Observe that we can make $\mathbb{E}Z = \mu(\mathcal{H}')$ as large as we wish by taking larger and larger subsets of $\mathcal{H}$. Using the Chernoff bound (Lemma 2.1) it thus follows that

$$\limsup_{n\to\infty} \mathbb{P}(N_n(\mathcal{H}) > K) \leq \lim_{n\to\infty} \mathbb{P}(N_n(\mathcal{H}') > K) = \mathbb{P}(Z > K) \leq \mathrm{e}^{-\mathbb{E}Z\cdot H(K/\mathbb{E}Z)},$$

(provided $\mathbb{E}Z > K$ is sufficiently large), where $H(x) := x\ln x - x + 1$. In particular, we can make $\mathbb{P}(Z > K)$ arbitrarily small by taking $\mathbb{E}Z = \mu(\mathcal{H}')$ sufficiently large. ■

Proof of Theorem 5.4: By the exponential formula, we also have $C(\rho) = \infty$. Since $n!/\operatorname{aut}(H)$ is exactly the number of labelled graphs isomorphic to H when $v(H) = n$, we have $\sum_{H\in\mathcal{UC}_n} \frac{1}{\operatorname{aut}(H)} = |\mathcal{C}_n|/n!$. With $\mu(H)$ as in Corollary 5.5, we get the following alternative expression for $C(\rho)$:

$$C(\rho) = \sum_n \frac{|\mathcal{C}_n|}{n!}\rho^n = \sum_{H\in\mathcal{UC}} \mu(H).$$

Since $C(\rho) = \infty$, we can define an infinite sequence $\mathcal{H}_1, \mathcal{H}_2, \dots$ of finite, disjoint subsets of $\mathcal{UC}$ with the property that $1000 \le \mu_i := \sum_{H \in \mathcal{H}_i} \mu(H) < 1001$ for each i. To see why the upper bound can be made to hold, keep in mind the definition of μ, and also that $\rho \le 1$ (by a result of Bernardi, Noy and Welsh [5]).

Let us set

$$E_i := \Big\{ \text{no component of } G_n \text{ is isomorphic to an element of } \mathcal{H}_i \Big\} \quad (i = 1, 2, \dots),$$

and

$$F_1 := E_1, \qquad F_i := E_1^c \cap \dots \cap E_{i-1}^c \cap E_i \quad (i = 2, 3, \dots).$$

The events F_i are clearly FO-expressible, and disjoint.

Let us now fix some index i. For $1 \le j \le i$, let N_j denote the number of components isomorphic to a graph in $\mathcal{H}_j$. Since the sum of independent Poisson-distributed random variables is again Poisson-distributed, it follows from Lemma 2.21 that

$$(N_1, \dots, N_i) \to_{\mathrm{TV}} (Z_1, \dots, Z_i),$$

where the Z_j are independent Poisson random variables with $\mathbb{E}Z_j = \mu_j$. We find that

$$\begin{array}{rcl} p_i & := & \lim_{n\to\infty} \mathbb{P}(F_i) \\ & = & \mathbb{P}(\mathrm{Po}(\mu_1) > 0) \cdots \mathbb{P}(\mathrm{Po}(\mu_{i-1}) > 0) \cdot \mathbb{P}(\mathrm{Po}(\mu_i) = 0) \\ & = & (1 - \mathrm{e}^{-\mu_1}) \cdots (1 - \mathrm{e}^{-\mu_{i-1}}) \cdot \mathrm{e}^{-\mu_i}. \end{array}$$

Observe that

$$1 \ge \sum_{j=1}^{i} p_j = 1 - \lim_{n\to\infty} \mathbb{P}(E_1^c \cap \dots \cap E_i^c) = 1 - (1 - \mathrm{e}^{-\mu_1}) \cdots (1 - \mathrm{e}^{-\mu_i}) \ge 1 - (1 - \mathrm{e}^{-1001})^i.$$

By sending $i \to \infty$ we see that $\sum_{i=1}^{\infty} p_i = 1$. Next, observe that since $\mu_i > 1000$ we have $\mathrm{e}^{-\mu_i} < 1 - \mathrm{e}^{-\mu_i}$. This also gives:

$$p_i = (1 - \mathrm{e}^{-\mu_1}) \cdots (1 - \mathrm{e}^{-\mu_{i-1}}) \cdot \mathrm{e}^{-\mu_i} \le (1 - \mathrm{e}^{-\mu_1}) \cdots (1 - \mathrm{e}^{-\mu_i}) = \sum_{j>i} p_j.$$

We can thus apply Lemma 2.2 to derive that

$$\left\{ \sum_{i \in A} p_i : A \subseteq \mathbb{N} \right\} = [0, 1].$$

For $A \subseteq \mathbb{N}$ a *finite* set, let us write $F_A := \bigcup_{i \in A} F_i$. Then F_A is clearly FO-expressible, and $\lim_{n\to\infty} \mathbb{P}(F_A) = \sum_{i \in A} p_i$. Let $\Phi \subseteq \mathrm{FO}$ be all corresponding FO-sentences (i.e., every $\varphi \in \Phi$ defines F_A for some finite A). We have:

$$\Big\{ \lim_{n\to\infty} \mathbb{P}(G_n \models \varphi) : \varphi \in \Phi \Big\} = \left\{ \sum_{i \in A} p_i : A \subseteq \mathbb{N} \text{ finite } \right\}.$$

Finally observe that for every $A \subseteq \mathbb{N}, \varepsilon > 0$ there is a *finite* $A' \subseteq A$ such that $|\sum_{i \in A} p_i - \sum_{i \in A'} p_i| < \varepsilon$. In other words, the limiting probabilities of Φ are dense in $[0, 1]$, as required. ■

5.5 The MSO-convergence law for forests of paths

For forests of paths one can easily see that $C(z) = z + \frac{1}{2} \sum_{n=2}^{\infty} z^n$ (since there is one path on $n = 1$ vertex, and there are $n!/2$ paths on n vertices for all $n \ge 2$). So the radius of convergence must be $\rho = 1$ and we have $C(\rho) = G(\rho) = \infty$. The following follows immediately from Corollary 5.5, since every path on at least two vertices has exactly two automorphisms and $\rho = 1$.

Corollary 5.6 *Let $\mathcal{F} \subseteq \mathcal{UC}$ be an infinite family of (unlabelled) paths, and let $K > 0$ be an arbitrary constant. W.h.p. the random forest of paths G_n contains at least K components isomorphic to members of $\mathcal{F}$.* ■

Recall that $\equiv_k^{\mathrm{MSO}}$ is an equivalence relation with finitely many classes. Let m denote the number of classes that contain at least one path, and let $\mathcal{C}_1, \dots, \mathcal{C}_m$ be a partition of all (unlabelled) paths according to their $\equiv_k^{\mathrm{MSO}}$-type. For each $1 \leq i \leq m$, let us pick an arbitrary representative $H_i \in \mathcal{C}_i$, and let us denote

$$\Gamma_k(a_1, \dots, a_m) := a_1 H_1 \cup \dots \cup a_m H_m.$$

That is, $\Gamma_k(a_1, \dots, a_m)$ is the vertex-disjoint union of a_i copies of H_i, for each i.

Proof of part (ii) of Theorem 1.10: Let $\varphi \in$ MSO be arbitrary and let k be its quantifier depth.

Recall that for $\mathcal{H} \subseteq \mathcal{C}$ we denote by $N_n(\mathcal{H})$ the number of components of G_n that are isomorphic to a member of $\mathcal{H}$. By Lemma 2.6 and the construction of $\Gamma_k(a_1, \dots, a_m)$ we have that

$$G_n \equiv_k^{\mathrm{MSO}} \Gamma_k(N_n(\mathcal{C}_1), \dots, N_n(\mathcal{C}_m)).$$

Let us assume (without loss of generality) that the classes $\mathcal{C}_1, \dots, \mathcal{C}_{m'}$ are finite and $\mathcal{C}_{m'+1}, \dots, \mathcal{C}_m$ are infinite for some $m' < m$; and let $a = a(k)$ be as provided by Lemma 2.7. We know from Corollary 5.6 that $N_n(\mathcal{C}_i) > K$ w.h.p. for all $i \geq m'$. Hence, applying Lemma 2.6 and Lemma 2.7, we find that

$$G_n \equiv_k^{\mathrm{MSO}} \Gamma'_k(N_n(\mathcal{C}_1), \dots, N_n(\mathcal{C}_{m'}), a, \dots, a) \quad \text{w.h.p.}$$

Now define

$$\mathcal{A} := \{(a_1, \dots, a_{m'}) \in \mathbb{Z}^{m'} : \Gamma_k(a_1, \dots, a_{m'}, a, \dots, a) \models \varphi\}.$$

Since a sum of independent Poisson random variables has again a Poisson distribution, it follows that $(N_n(\mathcal{C}_1), \dots, N_n(\mathcal{C}_{m'})) \to_{\mathrm{TV}} (Z_1, \dots, Z_{m'})$ where the Z_i are independent Poisson random variables with means $\mathbb{E}Z_i = \sum_{H \in \mathcal{C}_i} 1/\operatorname{aut}(H)$. It follows that

$$\lim_{n\to\infty} \mathbb{P}(G_n \models \varphi) = \lim_{n\to\infty} \mathbb{P}((N_n(\mathcal{C}_1), \dots, N_n(\mathcal{C}_{m'})) \in \mathcal{A}) = \mathbb{P}((Z_1, \dots, Z_{m'}) \in \mathcal{A}).$$

This proves the convergence law (since the rightmost expression does not depend on n). ■

5.6 The FO-convergence law for forests of caterpillars

For forests of caterpillars we have that $\rho \approx 0.567$, where ρ is the unique real root of $ze^z = 1$ ([7, Proposition 26]), and $G(\rho) = \infty$ ([7, Table 1]).

Recall that the spine of a caterpillar is the path consisting of all vertices of degree at least two. For G a forest of caterpillars, let $\mathrm{Long}_\ell(G)$ denote the union of all components whose spine has $> \ell$ vertices, let $\mathrm{Short}_{\ell,K}(G)$ denote the union of all components whose spine has at most ℓ vertices and whose degrees are all at most K, and $\mathrm{Bush}_{\ell,K}(G) := G \setminus (\mathrm{Long}_\ell(G) \cup \mathrm{Short}_{\ell,K}(G))$ denote the union of all remaining components. Before starting the proof of the convergence law for forests of caterpillars, we prove some lemmas on the subgraphs we just defined.

Lemma 5.7 *For every $\ell \in \mathbb{N}$ and $\varepsilon > 0$ there is a $K = K(\ell, \varepsilon)$ such that $\mathbb{P}(\mathrm{Bush}_{\ell,K}(G_n) \neq \emptyset) \leq \varepsilon$ for n sufficiently large.*

Proof: For $s \leq \ell$ and t arbitrary, let $E_{s,t}$ denote the event that G_n contains a caterpillar on $s + t$ vertices with s vertices on the spine. Observe that

$$\mathbb{P}(E_{s,t}) \leq \frac{\binom{n}{s+t} \cdot \binom{s+t}{s} \cdot \frac{s!}{2} \cdot s^t \cdot |\mathcal{G}_{n-(s+t)}|}{|\mathcal{G}_n|}.$$

(We choose $s+t$ vertices and construct a caterpillar of the required kind on them and a forest of caterpillars on the remaining vertices. This results in some over-counting, but that is fine for an upper bound. To construct the caterpillar on the $s+t$ chosen vertices, we first choose s vertices for the spine, we arrange these s vertices in one of $s!/2$ ways on a path and each of the remaining t vertices then chooses one of the vertices of the spine to attach to.)

Since the class of forests of caterpillars is smooth (Theorem 2.22), for every $\varepsilon > 0$ there is an $n_0 = n_0(\varepsilon)$ such that $(\rho - \varepsilon) \leq k|\mathcal{G}_{k-1}|/|\mathcal{G}_k| \leq (\rho + \varepsilon)$ for all $k \geq n_0$. It follows that if $n - (s+t) \geq n_0$ then

$$\begin{aligned} \mathbb{P}(E_{s,t}) &\leq \tfrac{(n)_{s+t}}{s!t!} \cdot \tfrac{s!}{2} \cdot s^t \cdot \tfrac{|\mathcal{G}_{n-(s+t)}|}{|\mathcal{G}_n|} \\ &\leq \tfrac{s^t}{2t!}(\rho+\varepsilon)^{s+t} = \tfrac{(\rho+\varepsilon)^s}{2} \cdot \tfrac{(s(\rho+\varepsilon))^t}{t!} \leq \tfrac{(s(\rho+\varepsilon))^t}{t!}. \end{aligned}$$

(The last inequality holds because we can assume without loss of generality that ε is small enough so that $(\rho+\varepsilon)^s/2 < 1$.) If, on the other hand, $n - (s+t) \leq n_0$ then

$$\begin{aligned} \mathbb{P}(E_{s,t}) &\leq \frac{n^{n_0} \cdot n^s \cdot \frac{s!}{2} \cdot s^n \cdot 2^{\binom{n_0}{2}}}{(n)_{n-n_0} \cdot (1/\rho - \varepsilon)^{n-n_0}} = \frac{n_0! \cdot (1/\rho - \varepsilon)^{n_0} \cdot n^{n_0} \cdot n^s \cdot \frac{s!}{2} \cdot s^n \cdot 2^{\binom{n_0}{2}}}{n! \cdot (1/\rho - \varepsilon)^n} \\ &= O\left(n^{n_0+s-1/2}\left(\tfrac{e \cdot s}{n \cdot (1/\rho - \varepsilon)}\right)^n\right) = o(1), \end{aligned}$$

as $n \to \infty$, where we have used that $1 \leq |\mathcal{G}_{n_0}| \leq 2^{\binom{n_0}{2}}$ in the first line and Stirling's approximation $n! \sim (n/e)^n \cdot \sqrt{2\pi n}$ in the second line. We thus have, for every T:

$$\mathbb{P}\left(\bigcup_{\substack{1 \leq s \leq \ell, \\ t \geq T}} E_{s,t}\right) \leq \sum_{\substack{1 \leq s \leq \ell, \\ t \geq T}} \frac{(s(\rho+\varepsilon))^t}{t!} + o(1).$$

Since $\sum_t \frac{(s(\rho+\varepsilon))^t}{t!}$ converges for every s, we can choose T such that $\mathbb{P}(\bigcup_{\substack{1 \leq s \leq \ell, \\ t \geq T}} E_{s,t}) < \varepsilon$ for sufficiently large n.

To conclude the proof of the lemma, we simply set $K(\ell, \varepsilon) = T + 2$ and observe that whenever $\text{Bush}_{\ell,K}(G_n) \neq \emptyset$ then $E_{s,t}$ must hold for some $s \leq \ell$ and $t \geq T$. ■

Lemma 5.8 *For every $k \in \mathbb{N}$ there is an $\ell = \ell(k)$ and a graph Q_k such that* $\text{Long}_\ell(G_n) \equiv_k^{\text{FO}} Q_k$ *w.h.p.*

The proof of this lemma makes use of an observation that we state as a separate lemma. Note that the isomorphism type of a caterpillar is described completely be a sequence of numbers $d_1, \ldots, d_\ell$ where ℓ is the number of vertices of the spine and d_i is the number of vertices of degree one attached to the i-th vertex of the spine. We shall call these numbers simply the *sequence* of the caterpillar. Let us say that a caterpillar has *begin sequence* $\overline{d} = (d_1, \ldots, d_\ell)$ (where always $d_1 \geq 1$) if its sequence either starts with $\overline{d}$ or it ends with $d_\ell, d_{\ell-1}, \ldots, d_1$. For every sequence $\overline{d}$ with $d_1 \geq 1$, let $\mathcal{H}_{\overline{d}} \subseteq \mathcal{UC}$ denote the set of unlabelled caterpillars with begin sequence $\overline{d}$.

Lemma 5.9 *For every sequence $\overline{d} = (d_1, \ldots, d_\ell)$ (with $d_1 \geq 1$), we have $\mu(\mathcal{H}_{\overline{d}}) = \infty$.*

Proof: Let $\mathcal{F} := \mathcal{UC} \setminus \mathcal{H}_{\overline{d}}$ be the set of those caterpillars that do not have begin sequence $\overline{d}$. Recall that $C(\rho) = \sum_{H \in \mathcal{UC}} \mu(H)$. Since $\mu(\mathcal{F}) + \mu(\mathcal{H}_{\overline{d}}) = C(\rho) = \infty$, we are done if $\mu(\mathcal{F}) < \infty$. Let us thus assume $\mu(\mathcal{F}) = \infty$.

For each $H \in \mathcal{F}$ let us define a $H' \in \mathcal{H}_{\overline{d}}$ by adding $\ell + \sum_{i=1}^{\ell} d_i$ vertices in the obvious manner. This clearly defines an injection mapping $\mathcal{F}$ to $\mathcal{H}_{\overline{d}}$. Also observe that for a caterpillar H with sequence $t_1, \ldots, t_k$, the number of automorphisms $\text{aut}(H)$ is either simply equal to $t_1! \cdots t_k!$ or to $t_1! \cdots t_k!/2$ (the latter case only occurs if the sequence is symmetric). Hence, it follows that

$$\mu(H') \geq \frac{\rho^{\ell+d_1+\cdots+d_\ell}}{2 \cdot d_1! \cdots d_\ell!} \cdot \mu(H),$$

for every $H \in \mathcal{F}$. It follows that $\mu(\mathcal{H}_{\overline{d}}) \geq \frac{\rho^{\ell+d_1+\cdots+d_\ell}}{2\cdot d_1!\cdots d_\ell!} \cdot \mu(\mathcal{F}) = \infty$, as required. ■

Combining this last lemma with Corollary 5.5, we immediately get that

Corollary 5.10 *For every sequence $\overline{d} = (d_1, \ldots, d_\ell)$ (with $d_1 \geq 1$) and every constant $K > 0$, G_n contains at least K components with beginsequence $\overline{d}$, w.h.p.*

Proof of Lemma 5.8: Let $k \in \mathbb{N}$ be arbitrary. Recall that, up to logical equivalence, there are only finitely many FO-sentences of quantifier depth at most k. Thus, by Gaifman's theorem (Theorem 2.8), there is a finite set $\mathcal{B} = \{\varphi_1, \ldots, \varphi_m\}$ such that every sentence of quantifier depth at most k is equivalent to a boolean combination of sentences in $\mathcal{B}$, and for each i we can write:

$$\varphi_i = \exists x_1, \ldots, x_{n_i} : \left(\bigwedge_{1 \leq a \leq n_i} \psi^{B(x_a, \ell_i)}(x_a) \right) \wedge \left(\bigwedge_{1 \leq a < b \leq n} \mathrm{dist}(x_a, x_b) > 2\ell_i \right).$$

Now let us set $\ell = 1000 \cdot \max_i \ell_i$. For each i, let us fix a caterpillar H_i with a spine of at least ℓ vertices that satisfies $H_i \models \exists x : \psi^{B(x,\ell_i)}(x)$, if such a caterpillar exists. Without loss of generality we can assume there exists a $m' < m$ such that the sought caterpillar H_i exists for all $i \leq m'$ and it does not exist for $i > m'$. Let us now set

$$Q_k := n_1 H_1 \cup \cdots \cup n_{m'} H_{m'}.$$

That is, Q_k is the vertex disjoint union of n_i copies of H_i, for each $i \leq m'$.

For $i > m'$ we have that both $Q_k \models \neg\varphi_i$ and $\mathrm{Long}_\ell(G_n) \models \neg\varphi_i$. (Since there is no caterpillar with a spine of at least ℓ vertices that satisfies $\exists x : \psi^{B(x,\ell_i)}(x)$.) On the other hand, for $i \leq m'$, we have that $Q_k \models \varphi$. Now note that, by choice of ℓ and H_i it is either the case that **1)** every caterpillar whose begin sequence is equal to the sequence of H_i satisfies $\exists x : \psi^{B(x,\ell_i)}(x)$, or **2)** every caterpillar whose begin sequence is equal to the reverse of the sequence of H_i satisfies $\exists x : \psi^{B(x,\ell_i)}(x)$. By Corollary 5.10, w.h.p., G_n contains at least n_i components with either begin sequence. Hence, w.h.p., $\mathrm{Long}_\ell(G_n) \models \varphi_i$.

We have seen that $Q_k \models \varphi_i$ if and only if $\mathrm{Long}_\ell(G_n) \models \varphi_i$ w.h.p. (for all $1 \leq i \leq m$). Since every FO-sentence of quantifier depth at most k is a boolean combination of $\varphi_1, \ldots, \varphi_m$, it follows that $Q_k \equiv_k^{\mathrm{FO}} \mathrm{Long}_\ell(G_n)$ w.h.p. ■

Proof of part (ii) of Theorem 1.11: Let us fix a $\varphi \in \mathrm{FO}$, let k be its quantifier depth and let ℓ, Q_k be as provided by Lemma 5.8. Let $\varepsilon > 0$ be arbitrary, and let $K = K(\ell, \varepsilon)$ be as provided by Lemma 5.7. Let $H_1, \ldots, H_m \in \mathcal{UC}$ be all (unlabelled) caterpillars whose spines have at most ℓ vertices and whose sequence has only numbers less than K; and let $N_n(H_i)$ denote the number of components of G_n isomorphic to H_i. By Lemma 2.21, we have that $(N(H_1), \ldots, N(H_m)) \to_{\mathrm{TV}} (Z_1, \ldots, Z_m)$, where the Z_i are independent Poisson random variables with means $\mathbb{E}Z_i = \mu(H_i) = \rho^{v(H_i)}/\operatorname{aut}(H_i)$. Let us set

$$\Lambda_k(a_1, \ldots, a_m) := a_1 H_1 \cup \cdots \cup a_m H_m \cup Q_k.$$

(That is, $\Lambda_k(a_1, \ldots, a_m)$ is the vertex disjoint union of Q_k with a_i copies of H_i for every i.) By Lemma 5.8, we have that $\mathrm{Long}_\ell(G_n) \equiv_k^{\mathrm{FO}} Q_k$ w.h.p. It follows that also $G_n \setminus \mathrm{Bush}_{\ell,K}(G_n) \equiv_k^{\mathrm{FO}} \Lambda(N(H_1), \ldots, N(H_m))$ w.h.p. Now let $\mathcal{A} := \{(a_1, \ldots, a_m) \in \mathbb{Z}^m : \Lambda_k(a_1, \ldots, a_m) \models \varphi\}$. We see that

$$\begin{array}{rcl} \mathbb{P}(G_n \models \varphi) & \leq & \mathbb{P}((N(H_1), \ldots, N(H_m)) \in \mathcal{A}) + \mathbb{P}(\mathrm{Bush}_{\ell,K}(G_n) \neq \emptyset) + \mathbb{P}(\mathrm{Long}_\ell(G_n) \not\equiv_k^{\mathrm{FO}} Q_k) \\ & \leq & \mathbb{P}((Z_1, \ldots, Z_m) \in \mathcal{A}) + \varepsilon + o(1), \end{array}$$

and, similarly

$$\begin{array}{rcl} \mathbb{P}(G_n \models \varphi) & \geq & \mathbb{P}((N(H_1), \ldots, N(H_m)) \in \mathcal{A}) - \mathbb{P}(\mathrm{Bush}_{\ell,K}(G_n) \neq \emptyset) - \mathbb{P}(\mathrm{Long}_\ell(G_n) \not\equiv_k^{\mathrm{FO}} Q_k) \\ & \geq & \mathbb{P}((Z_1, \ldots, Z_m) \in \mathcal{A}) - \varepsilon + o(1). \end{array}$$

These bounds show that $\limsup_{n\to\infty} \mathbb{P}(G_n \models \varphi)$ and $\liminf_{n\to\infty} \mathbb{P}(G_n \models \varphi)$ differ by at most 2ε. Sending $\varepsilon \downarrow 0$ then proves that the limit $\lim_{n\to\infty} \mathbb{P}(G_n \models \varphi)$ exists. ■

6 Discussion and further work

Here we mention some additional considerations and open questions that arise naturally from our work.

MSO limit laws for surfaces. While we were not able to extend our proofs of Theorem 1.3 and 1.4 to work for MSO, we believe that the results should generalize.

Conjecture 6.1 *Let $\mathcal{G}$ be the class of all graphs embeddable on a fixed surface S. Then $C_n \in_u \mathcal{C}_n$ obeys the* MSO*-zero-one law and $G_n \in_u \mathcal{G}_n$ obeys the* MSO*-convergence law.*

Let us mention that the proof of this conjecture is likely to be much more involved than the proofs of Theorems 1.3 and 1.4. The proof will probably have to take into account detailed information on the global structure of the largest component, and it may have to treat different surfaces separately. In MSO one can for instance express the property "G_n has an H-minor" (for any H) and by the results in [9], the random graph on a surface S will have the "correct" genus (i.e. G_n and C_n will not be embeddable on any "simpler" surface) with high probability. Hence, with high probability, at least one forbidden minor for embeddability on each simpler surface will have to occur. In particular, if the MSO-zero-one law/convergence law holds, then the value of the limiting probabilities of some MSO sentences will depend on the surface S. Let us also mention that no sensible analogue of Gaifman's locality theorem (Theorem 2.8) for MSO seems possible and that, related to this, in the MSO-Ehrenfeucht-Fraïssé game the set moves allow Spoiler to exploit global information. (For instance, if G is 4-colourable and H is not, then Spoiler can start by exhibiting a proper 4-colouring of G in four set-moves, and then catch Duplicator by either exhibiting a monochromatic edge or an uncoloured vertex in H. Since the chromatic number is a global characteristic – there are graphs that are locally tree-like yet have high chromatic number – this suggests that if we wish to prove the MSO-zero-one and/or convergence laws for random graphs on surfaces by considering the Ehrenfeucht-Fraïssé game, we may have to come up with a rather involved strategy for Duplicator.)

An attractive conjecture of Chapuy et al. [9] states that for every surface S, the random graph embeddable on S will have chromatic number equal to four with high probability. Since being k-colourable is expressible in MSO for every fixed k, establishing our conjecture above can be seen as a step in the direction of the Chapuy et al. conjecture. In [9] it was already shown that the chromatic number is $\in \{4, 5\}$ with high probability. Proving the MSO-zero-one law will imply that the chromatic number of the random graph is either four w.h.p. or five w.h.p. (as opposed to some probability mass being on 4 and some on 5, or oscillating between the two values).

The limiting probabilities. In all the cases where we have established the convergence law in this paper, it turned out that the closure of the set of limiting probabilities is either simply $\{0, 1\}$ or a union of finitely many closed intervals. A natural question is whether there are choices of a model of random graphs for which the convergence law holds, but we end up with a more exotic set. The answers happens to be yes. For instance, if in the binomial random graph model $G(n, p)$ we take $p = \frac{\ln n}{n} + \frac{c}{n}$ then the FO-convergence law holds and we get (see [38], Section 3.6.4) that

$$\mathrm{cl}\left(\left\{\lim_{n\to\infty} \mathbb{P}(G(n,p) \models \varphi) : \varphi \in \mathrm{FO}\right\}\right) = \left\{\sum_{i\in A} p_i : A \subseteq \{0\} \cup \mathbb{N}\right\}, \tag{13}$$

where $p_i := \mu^i e^{-\mu}/i!$ with $\mu = e^{-c}$. Observe that $p_i > \sum_{j>i} p_j$ for all sufficiently large i. From this it follows that the right hand side of (13) is homeomorphic to the Cantor set (see [35]).

Returning to random graphs from minor-closed classes, let us recall that for every addable, minor-closed class we have that $G(\rho) \leq e^{1/2}$ by a result of Addario et al. [1] and independently Kang and Panagiotou [23]. In the notation of Section 4 we have $p_1 = 1/G(\rho)$, so that this gives that $p_1 > 1 - p_1$. Corollary 2.3 allows us to deduce from this that there is at least one "gap" in the closure of the limiting probabilities. In the case of forests there are in fact three gaps in total. We believe that every addable, minor-closed class has at least three gaps, and moreover that the reason is the following.

Conjecture 6.2 *If $\mathcal{G}$ is an addable minor-closed class, $G(z)$ its exponential generating function and ρ the radius of convergence, then*

$$G(\rho) < 1 + 2\rho.$$

In the notation of Section 4 we have $p_1 = 1/G(\rho)$ and $p_2 = \rho/G(\rho)$. So the conjecture is equivalent to $p_2 > 1 - p_1 - p_2$, which together with Corollary 2.3 will indeed imply that the closure of the set of limiting probabilities consists of at least four intervals (that is, there are at least three gaps). The difficulty lies on relating the values of ρ and $G(\rho)$.

MSO-convergence law for smooth and decomposable graph classes. We were able to show that for forests of paths, the MSO-convergence law holds. For forests of caterpillars we were only able to show the FO-convergence law, but we believe the MSO-convergence law should hold too. What is more, we conjecture that this should be true in general for every minor-closed class that is both smooth and decomposable.

Conjecture 6.3 *For every decomposable, smooth, minor-closed class, the MSO-convergence law holds.*

In the proof of the MSO-convergence law for forests of paths, we considered the partition $\mathcal{C}_1, \dots, \mathcal{C}_m$ of all unlabelled paths $\mathcal{UC}$ into $\equiv_k^{\mathrm{MSO}}$-equivalence classes. We used that whenever $|\mathcal{C}_i| = \infty$, then $\mathcal{N}_n(\mathcal{C}_i)$, the number of components isomorphic to elements of $\mathcal{C}_i$, will grow without bounds, whereas if $|\mathcal{C}_i| < \infty$, then $\mathcal{N}_n(\mathcal{C}_i)$ will tend to a Poisson distribution. This essentially relied on the fact that all paths on at least two vertices have precisely two automorphisms.

One would hope that for other smooth and decomposable minor-closed classes some variation of the proof for forests of paths, i.e. considering the number of components belonging to each $\equiv_k^{\mathrm{MSO}}$-equivalence class separately, might lead to a proof of our conjecture. For paths it was very easy to show a dichotomy between $\mathcal{N}_n(\mathcal{C}_i)$ growing without bounds or $\mathcal{N}_n(\mathcal{C}_i)$ following a Poisson law. For general smooth, decomposable, minor-closed classes one might still expect a similar dichotomy although the proof, even for forests of caterpillars, is likely to be more technically involved than in the case of forests of paths.

Extensions to the 2-addable and rooted case. Possible extensions of our results are to classes of graphs with higher connectivity properties. Call a minor-closed class $\mathcal{G}$ *2-addable* if it is addable and it is closed under the operation of gluing two graphs through a common edge. This is equivalent to the fact that the minimal excluded minors are 3-connected. An MSO-zero-one law should hold for *2-connected* graphs in $\mathcal{G}$, by proving an analogous of Theorem 2.12 for pendant copies of a 2-connected graph overlapping with the host graph in exactly one specified edge. Moreover, we believe an MSO-convergence law should hold for *rooted* connected planar graphs, adapting the proof by Woods [39] for rooted trees (the zero-one law does not hold since, for example, the probability that the root vertex has degree k tends to a constant strictly between 0 and 1). We leave both problems for future research.

Unlabelled graphs. In enumerative combinatorics, unlabelled objects are typically much harder to deal with than labelled ones. We strongly believe that our results on addable classes will extend to the unlabelled case.

Conjecture 6.4 *Let $\mathcal{G}$ be an addable, minor-closed class and let $\mathcal{UG}$ be the corresponding collection of unlabelled graphs. The MSO-zero-one law holds for $C_n \in_u \mathcal{UC}_n$, the random* connected*, unlabelled graph from $\mathcal{G}$. The MSO-convergence law holds for $G_n \in_u \mathcal{UG}_n$.*

The previous conjecture would follow with the same proofs as in Theorems 1.1 and 1.2, provided the analogue of Theorem 2.14 on pendant copies holds. It is believed that this is the case, but so far it has been proved only for so-called subcritical classes [12]. These include forests, outerplanar and series-parallel graphs, but not the class of planar graphs.

Analogues of the Rado graph. Although we are do not formulate any research question in this direction, we cannot resist mentioning some of our thoughts concerning analogues of the Rado graph, a beautiful mathematical object that is associated with the FO-zero-one law for the binomial random graph $G(n, 1/2)$. If $\mathcal{T}$ is the set of all FO-sentences φ such that $\lim_{n\to\infty} \mathbb{P}(G(n, 1/2) \models \varphi) = 1$, then as it happens there is (up to isomorphism) exactly one countable graph that satisfies all sentences in $\mathcal{T}$, namely the Rado graph. The Rado graph has several remarkable properties, and surprisingly, it is connected to seemingly far-removed branches of mathematics such as number theory and topology. See [8] and the references therein for more background on the Rado graph.

One might wonder whether, for the cases where we have proved the zero-one law, a similar object might exist. It follows in fact from general arguments from logic that there will always be at least one countable graph that satisfies every sentence that has limiting probability one. What is more, carefully re-examining the proof of Theorem 3.1, we find that we can construct such a graph by identifying the roots of $M_1, M_2, \dots$ with M_k as in Theorem 3.1. One might hope that, similarly to the case of the Rado graph, our graph is the unique (up to isomorphism) countable graph that satisfies precisely those MSO sentences that have limiting probability one. By some straightforward variations on the construction (for instance by not attaching the consecutive M_k-s directly to the root, but rather by hanging them from the root using paths of varying length) we can however produce an *uncountable* family of non-isomorphic graphs with this property.

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