A threshold for the Maker-Breaker clique game*

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Abstract

We study the Maker-Breaker k-clique game played on the edge set of the random graph $G(n,p)$. In this game, two players, Maker and Breaker, alternately claim unclaimed edges of $G(n,p)$, until all the edges are claimed. Maker wins if he claims all the edges of a k-clique; Breaker wins otherwise. We determine that the threshold for the graph property that Maker can win this game is at $n^{-\frac{2}{k+1}}$, for all $k > 3$, thus proving a conjecture from [20]. More precisely, we conclude that there exist constants $c, C > 0$ such that when $p > Cn^{-\frac{2}{k+1}}$ the game is Maker's win a.a.s., and when $p < cn^{-\frac{2}{k+1}}$ it is Breaker's win a.a.s.

For the triangle game, when $k = 3$, we give a more precise result, describing the hitting time of Maker's win in the random graph process. We show that, with high probability, Maker can win the triangle game exactly at the time when a copy of K_5 with one edge removed appears in the random graph process. As a consequence, we are able to give an expression for the limiting probability of Maker's win in the triangle game played on the edge set of $G(n,p)$.

1 Introduction

Let X be a finite set and let $\mathcal{F} \subseteq 2^X$ be a family of subsets of X. In the positional game $(X, \mathcal{F})$, two players take turns in claiming one previously unclaimed element of X. The set X is called the "board", and the members of $\mathcal{F}$ are referred to as the "winning sets". In a *Maker-Breaker* positional game, the two players are called Maker and Breaker. Maker wins the game if he occupies all elements of some winning set; Breaker wins otherwise. We will assume that Maker starts the game. A game $(X, \mathcal{F})$ is said to be a *Maker's win* if Maker has a strategy that ensures his win against any strategy of Breaker; otherwise it is a *Breaker's win.* Note that $\mathcal{F}$ alone determines whether the game is Maker's win or Breaker's win.

A well-studied class of positional games are the *games on graphs*, where the board is the set of edges of a graph. The winning sets in this case are usually representatives of some graph theoretic structure. Research in this area was initiated by Lehman [18], who studied the connectivity game, a generalization of the well-known Shannon switching

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game, where Maker's goal is to claim a spanning connected graph by the end of the game. We denote the game by $(E(K_n), \mathcal{T})$. Another important game is the Hamilton cycle game $(E(K_n), \mathcal{H})$, where $\mathcal{H} = \mathcal{H}_n$ consists of the edge sets of all Hamilton cycles of K_n.

In the clique game the winning sets are the edge sets of all k-cliques, for a fixed integer $k \geq 3$. We denote this game with $(E(K_n), \mathcal{K}_k)$. Note that the size of the winning sets is fixed and does not depend on n, which distinguishes it from the connectivity game and the Hamilton cycle game. A simple Ramsey argument coupled with the strategy stealing argument (see [3] for details) ensures Maker's win if n is large. Beck [2] determined the largest clique Maker can claim on $E(K_n)$ with remarkable precision.

All three games that we introduced are straightforward Maker's wins when n is large enough. This is however not the end of the story. We present two general approaches to even out the odds, giving Breaker more power – *biased games* and *random games.*

Biased games. Biased games are a widely studied generalization of positional games, introduced by Chvátal and Erdős in [9]. Given two positive integers a and b and a positional game $(X, \mathcal{F})$, in the *biased* $(a : b)$ *game* Maker claims a previously unclaimed elements of the board in one move, while Breaker claims b previously unclaimed elements. The rules determining the outcome remain the same. The games we introduced initially are $(1 : 1)$ games, also referred to as *unbiased games.*

Now, if an unbiased game $(X, \mathcal{F})$ is a Maker's win, we choose to play the same game with $(1 : b)$ bias, increasing b until Breaker starts winning. Formally, we want to answer the following question: What is the largest integer $b_{\mathcal{F}}$ for which Maker can win the biased $(1 : b_{\mathcal{F}})$ game? This value is called the *threshold bias* of $\mathcal{F}$.

For the connectivity game, it was shown by Chvátal and Erdős [9] and Gebauer and Szabó [11] that the threshold bias is $b_{\mathcal{T}} = (1 + o(1))\frac{n}{\log n}$. The result of Krivelevich [16] gives the leading term of the threshold bias for the Hamilton cycle game, $b_{\mathcal{H}} = (1 + o(1))\frac{n}{\log n}$. In the k-clique game, Bednarska and Łuczak [4] found the order of the threshold bias, $b_{\mathcal{K}_k} = \Theta(n^{\frac{2}{k+1}})$. Determining the leading constant inside the $\Theta(.)$ remains an open problem that appears to be very challenging. Some lower and upper bounds can be derived from the calculations in [4], and for $k = 3$ better bounds are obtained in [1].

Random games. An alternative way to give Breaker more power in a positional game, introduced by the second author and Szabó in [20], is to randomly thin out the board before the game starts, thus eliminating some of the winning sets.

For games on graphs, given a game $\mathcal{F}$ that is Maker's win when played on $E(K_n)$, we want to find the *threshold probability* $p_{\mathcal{F}}$ so that, if the game is played on $E(G(n, p))$, an almost sure Maker's win turns into an almost sure Breaker's win, that is,

$$Pr[\mathcal{F} \text{ played on } E(G(n,p)) \text{ is Breaker's win}] \to 1 \text{ for } p = o(p_{\mathcal{F}}),$$

and

$$Pr[\mathcal{F} \text{ played on } E(G(n,p)) \text{ is Maker's win}] \to 1 \text{ for } p = \omega(p_{\mathcal{F}}),$$

when $n \to \infty$. Such a threshold $p_{\mathcal{F}}$ exists by a general result of Bollobás and Thomason [8], as "being Maker's win" is clearly a monotone increasing graph property.

The threshold probability for the connectivity game was determined to be $\frac{\log n}{n}$ in [20], and shown to be sharp. As for the Hamilton cycle game, the order of magnitude of the threshold was given in [19]. Using a different approach, it was proven in [13] that the threshold is $\frac{\log n}{n}$ and it is sharp. Finally, as a consequence of a hitting time result, Ben-Shimon et al. [5] closed this question by giving a very precise description of the low order terms of the limiting probability.

Moving to the clique game, it was shown in [20] that for every $k \geq 4$ and every $\varepsilon > 0$ we have

$$n^{-\frac{2}{k+1}-\varepsilon} \leq p_{\mathcal{K}_k} \leq n^{-\frac{2}{k+1}}.$$

Moreover, it was proved that there exist a constant $C > 0$ such that for $p \geq Cn^{-\frac{2}{k+1}}$ Maker wins the k-clique game on $G(n,p)$ a.a.s. The threshold for the triangle game was determined to be $p_{\mathcal{K}_3} = n^{-\frac{5}{9}}$, showing that the behavior of the triangle game is different from the k-clique game for $k \geq 4$, as $\frac{9}{5} < \frac{3+1}{2} = 2$.

Our main result is the following theorem. It gives a lower bound on the threshold for the k-clique game, when $k \geq 4$, which matches the upper bound from [20] up to the leading constant.

Theorem 1.1 *Let $k \geq 4$. There exists a constant $c > 0$ such that for $p \leq cn^{-\frac{2}{k+1}}$ Breaker wins the Maker-Breaker k-clique game played on the edge set of $G(n,p)$ a.a.s.*

The threshold probability for the k-clique game for $k \geq 4$ was conjectured to be $p_{\mathcal{K}_k} = n^{-\frac{2}{k+1}}$ in [20]. The previous theorem resolves this conjecture in the affirmative. Summing up the results of Theorem 1.1 and Theorem 19 from [20], we now have the following.

Corollary 1.2 *Let $k \geq 4$ and consider the Maker-Breaker k-clique game on the edge set of $G(n,p)$. There exist constants $c, C > 0$ such that the following hold:*

- **(i)** *If $p \geq Cn^{-\frac{2}{k+1}}$, then Maker wins a.a.s.;*
- **(ii)** *If $p \leq cn^{-\frac{2}{k+1}}$, then Breaker wins a.a.s.*

A result of this type is sometimes called a "semi-sharp threshold" in the random graphs literature.

Hitting time of Maker's win. We look at the same collection of positional games on graphs, now played in a slightly different random setting. Let V be a set of cardinality n, and let π be a permutation of the set $\binom{V}{2}$. If by G_i we denote the graph on the vertex set V whose edges are the first i edges in the permutation π, $G_i = (V, \pi^{-1}([i]))$, then we say that $\tilde{G} = \{G_i\}_{i=0}^{\binom{n}{2}}$ is a *graph process.* Given a monotone increasing graph property $\mathcal{P}$ and a graph process $\tilde{G}$, we define the hitting time of $\mathcal{P}$ with $\tau(\tilde{G};\mathcal{P}) = \min\{t : G_t \in \mathcal{P}\}$. If π is chosen uniformly at random from the set of all permutations of the set $\binom{V}{2}$, we say that $\tilde{G}$ is a *random graph process.* Such processes are closely related to the model of random graph we described above, see, e.g., [14].

Given a positional game, our general goal is to describe the hitting time of the graph property "Maker's win" in a typical graph process. For a game $\mathcal{F}$, by $\mathcal{M}_{\mathcal{F}}$ we denote the graph property "Maker wins $\mathcal{F}$". It was shown in [20] that in the connectivity game (with the technical assumption that Breaker is the first to play), for a random graph process $\tilde{G}$, we have $\tau(\tilde{G};\mathcal{M}_{\mathcal{T}}) = \tau(\tilde{G};\delta_2)$, where δ_ℓ is the graph property "minimum degree at least ℓ". Recently, Ben-Shimon et al. [5] resolved the same question for the Hamilton cycle game, obtaining $\tau(\tilde{G};\mathcal{M}_{\mathcal{H}}) = \tau(\tilde{G};\delta_4)$. Note that inequality in one direction for both of these equalities holds trivially.

Moving on to the clique game, we denote the property "the graph contains $K_5 - e$ as a subgraph" with $\mathcal{G}_{\mathcal{K}_5^-}$. We are able to show the following hitting time result for Maker's win in the triangle game.

Theorem 1.3 *For a random graph process $\tilde{G}$, the hitting time for Maker's win in the triangle game is asymptotiaclly almost surely the same as the hitting time for appearance of $K_5 - e$, i.e., $\tau(\tilde{G}; \mathcal{M}_{\mathcal{K}_3}) = \tau(\tilde{G}; \mathcal{G}_{\mathcal{K}_5^-})$ a.a.s.*

By considering the number of copies of $K_5 - e$, Theorem 1.3 together with the result proved independently by Bollobás [6] and Karoński and Ruciński [15] (also occurring as Theorem 3.19 in [14]) we are able to give a precise expression for the probability that Maker wins the triangle game on $G(n, p)$.

Corollary 1.4 *Let $p = p(n)$ be an arbitrary sequence of numbers $\in [0, 1]$ and let us write $x = x(n) = p \cdot n^{\frac{2}{k+1}}$. Then*

$$\lim_{n\to\infty} Pr[\text{Maker wins the triangle game on } G(n,p)] = \begin{cases} 0 & \text{if } x \to 0, \\ 1 - e^{-c^5/3} & \text{if } x \to c \in \mathbb{R}, \\ 1 & \text{if } x \to \infty. \end{cases}$$

1.1 Preliminaries

Throughout the paper, log stands for the natural logarithm. The notation $\text{Po}(\lambda)$ stands for the Poisson distribution with parameter λ. Our graph-theoretic notation is standard and follows the one from [21]. Given graphs G_1 and G_2, we say that $G_1 \cup G_2$ is the graph with the vertex set $V(G_1) \cup V(G_2)$ and the edge set $E(G_1) \cup E(G_2)$, and the graph $G_1 \cap G_2$ has the vertex set $V(G_1) \cap V(G_2)$ and the edge set $E(G_1) \cap E(G_2)$. A graph G is said to be an ℓ-degenerate graph, if there is an ordering of the vertices of G such that every vertex has at most ℓ neighbors that precede it in this ordering. The *density* of G is defined as $d(G) = \frac{e(G)}{v(G)}$, and the *maximum density* of G is defined as $m(G) = \max_{H \subseteq G} d(H)$. We say a graph is *strictly balanced* if $d(G) > d(H)$ for all proper subgraphs H of G. We will use the following theorem on the occurrence of small subgraphs in $G(n, p)$, due to Bollobás [6].

Theorem 1.5 ([6]) *Let G be an arbitrary graph with at least one edge. Then*

$$\lim_{n\to\infty} Pr[G(n,p) \text{ contains a copy of } G] = \begin{cases} 0 & \text{if } p \ll n^{-1/m(G)}, \\ 1 & \text{if } p \gg n^{-1/m(G)}. \end{cases}$$

We will make use of the following observation.

Lemma 1.6 *For every $k \geq 4$, Breaker wins the k-clique game played on the edge set of K_{k+2}, even if Maker starts.*

We give a quick sketch of the proof. Breaker wins the 4-clique game on K_6, which follows from a simple case analysis. An inductive argument shows that Breaker wins for larger k: Suppose Breaker wins the k-clique game on K_{k+2}. To win the $(k + 1)$-clique game on K_{k+3} he simply picks $k + 2$ vertices arbitrarily and makes sure that Maker does not claim a k-clique on them.

We will also need a result by Hales and Jewett [12]. The following lemma is a rewording of Lemma 7 in [12].

Lemma 1.7 ([12]) *Consider the unbiased Maker-Breaker game with board X and winning sets $\mathcal{F} \subseteq 2^X$. If there is an ℓ such that $|F| \geq \ell$ for all $F \in \mathcal{F}$ and every $x \in X$ is contained in at most $\ell/2$ elements of $\mathcal{F}$, then Breaker wins the game.*

2 The k-clique game on the random graph, for $k \geq 4$

Before we start doing probability theory, we will give a number of deterministic observations we need for the analysis of the Maker-Breaker clique game on $G(n, p)$.

Lemma 2.1 *Let G be a graph with maximum density at most $\frac{k+1}{2}$, where $k \geq 4$. Then Breaker wins the k-clique game on G, even if Maker starts.*

Proof. Let $k \geq 4$ be arbitrary. Aiming for a contradiction, let us assume that G is a minimal counterexample to the statement of the lemma. That is, G has maximum density at most $\frac{k+1}{2}$, and G itself is a win for Maker, while every proper subgraph of G is a win for Breaker.

We now claim that G must be $(k+1)$-edge-connected. To see this, suppose for a contradiction that there is a nontrivial partition A_1, A_2 of the vertex set $V(G)$ such that there are at most k edges between A_1 and A_2. By the minimality of G, there is a winning strategy for Breaker on $G_1 := G[A_1]$, and a winning strategy for Breaker on $G_2 := G[A_2]$. Breaker can then employ the following strategy: whenever Maker plays on an edge between two vertices in A_i he responds according to his winning strategy for G_i $(i = 1, 2)$, and if Maker claims an edge between A_1 and A_2, then Breaker claims another edge between A_1 and A_2 (if possible; otherwise he claims an arbitrary free edge). This way, by the end of the game Maker's graph contains at most $k-2$ edges between A_1 and A_2. Hence Maker cannot have claimed a k-clique that contains vertices of both A_1 and A_2. Maker also cannot have claimed a k-clique with all vertices in A_i $(i = 1, 2)$. Thus, Breaker can win the game on G, contradicting choice of G as a minimal counterexample. It must thus be that G is $(k+1)$-edge-connected as claimed.

Since G is $(k+1)$-edge-connected, every vertex of G must have degree $d(v) \geq k+1$. Next, we claim that G is in fact $(k+1)$-regular. To see this, observe that if there were some vertex v with $d(v) > k+1$ then, since the sum of the degrees is at most $(k+1) \cdot v(G)$ by the assumption on the maximum density, there would also be a vertex u with $d(u) < k+1$. But this would contradict that G is $(k+1)$-edge connected! Thus, G is $(k+1)$-regular as claimed.

By Lemma 1.6 above, G cannot be the $(k+2)$-clique. It follows that G has at least $k+3$ vertices.

Next, we claim that every edge of G is contained in at most $k-1$ cliques. To see this, let us fix an arbitrary edge $uv \in E(G)$, and let A denote the set of common neighbors of u and v. That is, $A := \{z \in V(G) : uz, vz \in E(G)\}$. Observe that, since $d(u) = d(v) = k+1$, we have that $|A| \leq k$. Let $B := A \cup \{u, v\}$. Since G is $(k+1)$-connected and not a $(k+2)$-clique there must be at least $k+1$ edges between B and $V(G) \setminus B$. Therefore the graph $G[B]$ induced on B must miss at least $\lceil (k+1)/2 \rceil \geq 3$ edges. By definition of A these missing edges are never incident with either u or v.

First suppose that three missing edges form a triangle. That is, there are three distinct vertices $a_1, a_2, a_3 \in A$ such that all three edges a_1a_2, a_1a_3, a_2, a_3 are missing. In that case any k-clique containing u, v contains at most one of these three vertices. Hence there are only three potential k-cliques containing u and v, namely $B \setminus \{a_2, a_3\}, B \setminus \{a_1, a_2\}, B \setminus \{a_1, a_2\}$. Note that $3 \leq k-1$, so the claim holds in this case.

Next, suppose that three missing edges form a $K_{1,3}$, i.e., there are vertices $a_1, a_2, a_3, a_4 \in A$ such that the edges a_1a_2, a_1a_3, a_1a_4 are missing. In this case any k-clique containing u, v and a_1 cannot contain any of a_2, a_3, a_4. But $|B \setminus \{a_2, a_3, a_4\}| < k$, so in fact every

k-clique that contains u, v misses a_1. Hence there are at most $\binom{k-1}{k-2} = k-1$ many k-cliques containing u, v.

Next, suppose that three missing edges form a path of length three. That is, there are distinct vertices a_1, a_2, a_3, a_4 such that a_1a_2, a_2a_3, a_3a_4 are all missing. Observe that any k-clique containing u, v must contain at least two of a_1, a_2, a_3, a_4 (since $|B \setminus \{a_1, \ldots, a_4\}| \leq k-2$) and it is not possible to contain three or more of these vertices. Hence, in this case the only potential k-cliques containing u, v are $B \setminus \{a_2, a_3\}, B \setminus \{a_2, a_4\}, B \setminus \{a_1, a_3\}$.

Let us now suppose that the three missing edges form the (vertex-) disjoint union of a path of length two and a path of length one. In other words, there are $a_1, \ldots, a_5 \in A$ such that a_1a_2, a_2a_3, a_4a_5 are missing. In this case the only possible k-cliques are $B \setminus \{a_2, a_4\}$ and $B \setminus \{a_2, a_5\}$.

Finally, let us observe that if none of the above cases occur then there are three missing edges that do not share endpoints, but in this case there can be no k-clique that contains u and v.

It follows that every edge $uv \in E(G)$ is contained in at most $k-1$ cliques, as claimed. Since $k-1 \leq \binom{k}{2}/2$ for all $k \geq 4$ it now follows immediately from Lemma 1.7 that Breaker has a winning strategy, contradicting our choice of G as a minimal counterexample for Maker's win. ■

For the remainder of this section, $k \geq 4$ will be a fixed integer. We describe an auxiliary graph that will play a role in our analysis of the clique game on $G(n, p)$. For G an arbitrary graph, let $\mathcal{F}^{(0)}(G) = \{C_1, \ldots, C_N\}$ denote the family of all k-cliques in G. We will dynamically change this collection of graphs by repeatedly applying the following rule.

Merging rule: If there are two graphs $G_1, G_2 \in \mathcal{F}^{(i)}$ such that $|V(G_1) \cap V(G_2)| \geq 3$, then we set $\mathcal{F}^{(i+1)} := \{G_1 \cup G_2\} \cup (\mathcal{F}^{(i)} \setminus \{G_1, G_2\})$.

After some number T of iterations, the rule can no longer be applied. For this choice of T, let us write $\mathcal{F}(G) := \mathcal{F}^{(T)}$. Observe that the final collection of graphs $\mathcal{F}$ does not depend on the order in which we apply the merging rule. Let $\mathcal{H}(G)$ denote the graph with vertex set $\mathcal{F}$, and an edge G_1G_2 if $|E(G_1) \cap E(G_2)| \neq 0$. (Note that in fact, by definition of $\mathcal{F}$, for every distinct $H, H' \in \mathcal{F}$ we have $|E(H) \cap E(H')| \in \{0, 1\}$.)

We note that in the study of the so-called clique percolation on random graphs, see [7], a similar structure of overlapping k-cliques is observed. However, since in our case cliques can simultaneously have different size of intersection with other cliques, we are unable to directly apply any results from this area of research.

If $e, f \in E(G)$ are two edges of G, then we call a path $G_1, \ldots, G_N$ in $\mathcal{H}(G)$ an (e, f)-path if $e \in G_1$ and $f \in G_N$.

Let us now list a number of properties a graph G can have.

- **(A)** Every subgraph $H \subseteq G$ with at most $v(H) \leq k^{1000}$ vertices has density at most $d(H) \leq \frac{k+1}{2}$;
- **(B)** Every $H \in V(\mathcal{H}(G))$ has at most $v(H) \leq 9k$ vertices;
- **(C)** Every connected subgraph $\mathcal{T} \subseteq \mathcal{H}(G)$ with at most $v(\mathcal{T}) \leq k^{997}$ vertices has at most 8 vertices that are not k-cliques;
- **(D)** For every connected subgraph $\mathcal{T} \subseteq \mathcal{H}(G)$ with $v(\mathcal{T}) \leq k^{997}$ there are at most k^{16} pairs of distinct edges $e \neq f \in E(\bigcup \mathcal{T})$ such that there is an (e, f)-path in $\mathcal{H}(G) \setminus \mathcal{T}$;

(E) For any $G_1, G_2 \in V(\mathcal{H}(G))$ with $\mathrm{dist}_{\mathcal{H}}(G_1, G_2) \leq k^{995}$, let $\mathcal{T}$ be a subgraph of $\mathcal{H}(G)$ induced on

$$\{H \in V(\mathcal{H}(G)) : \mathrm{dist}_{\mathcal{H}}(G_1, H) + \mathrm{dist}_{\mathcal{H}}(H, G_2) = \mathrm{dist}_{\mathcal{H}}(G_1, G_2)\}.$$

Then $v(\mathcal{T}) \leq 9 \cdot \mathrm{dist}_{\mathcal{H}}(G_1, G_2)$.

These properties are desirable from Breaker's perspective as the following lemma shows.

Lemma 2.2 *Let G be a graph that has properties* **(A)**-**(E)***. Then Breaker wins the k-clique game on G.*

Proof. Aiming for a contradiction, assume that Maker wins on G. Obviously, edges of G not participating in any k-clique make no difference in the game, so Maker can also win on $\bigcup \mathcal{H}(G)$. Let $\mathcal{T} \subseteq \mathcal{H}(G)$ be a minimal subgraph of $\mathcal{H}(G)$ such that Maker wins on $\bigcup \mathcal{T}$. That is, Maker wins on $\bigcup \mathcal{T}$, but Breaker wins on $\bigcup \mathcal{T}'$ for any proper subgraph $\mathcal{T}' \subseteq \mathcal{T}$.

If $\mathcal{T}$ were disconnected then, due to the minimality of $\mathcal{T}$, Breaker could win the game on each of the graphs corresponding to the vertex set of a component of $\mathcal{T}$. But, since all those games are disjoint, both in board and winning sets, Breaker could then also win the game on $\bigcup \mathcal{T}$, a contradiction. Hence $\mathcal{T}$ is connected.

Let us say that an edge $e \in E(\bigcup \mathcal{T})$ is *exposed* if there is exactly one $H \in V(\mathcal{T})$ such that $e \in E(H)$.

Next, observe that for every *k-clique* $C \in V(\mathcal{T})$ at most one edge $e \in E(C)$ is exposed. To see this, suppose there is a k-clique $C \in V(\mathcal{T})$ with two edges $e_1, e_2 \in E(C)$ exposed. Due to the minimality of $\mathcal{T}$, Breaker has a winning strategy for the game played on $G' := \bigcup(\mathcal{T} \setminus C)$. If Breaker, now playing on G', uses this strategy to respond to all Maker's moves on G', and additionally pairs edges e_1 and e_2 (meaning that when Maker claims one of them, Breaker claims the other in the following move), Maker will not be able to claim a clique, a contradiction. Hence, for every *k-clique* in $\mathcal{T}$ at most one of its edges is exposed.

We proceed by making a number of intermediate observations.

Claim 1 $\mathrm{diam}(\mathcal{T}) \leq k^{30}$*.*

Proof of Claim 1: Let us assume that $\mathrm{diam}(\mathcal{T}) \geq k^{30} + 1$, and let $H, H' \in V(\mathcal{T})$ be such that they realize the diameter. I.e. there is a path $G_0, \ldots, G_D$ in $\mathcal{T}$ such that $G_0 = H, G_D = H'$ and $D = \mathrm{diam}(\mathcal{T})$ and there are no shorter paths between H, H'.

Let $\mathcal{P}$ denote the subpath $\mathcal{P} := G_{D-k^{30}}, \ldots, G_D$. Let $J \subseteq \{D - k^{30}, \ldots, D\}$ denote the set of indices for which either G_i is not a k-clique or there is an edge $e \in E(G_i)$ and an edge $f \in E(\bigcup \mathcal{P}) \setminus \{e\}$ such that there exists an (e, f)-path in $\mathcal{H}(G) \setminus \mathcal{P}$. By property **(C)** and **(D)** we have $|J| \leq 8 + 4k^{16}$ (the 4 comes from the fact that for each pair (e, f) each of the two edges can occur in two vertices G_i, G_{i+1} of the path $\mathcal{P}$). Let us set $I := \{D - k^{30}, \ldots, D\} \setminus J$. Observe that

$$|I| \geq k^{30} + 1 - (8 + 4k^{16}) > k^{29}. \tag{1}$$

For each $i \in I$ let us fix an edge $e_i \in E(G_i)$ that is not exposed and is not contained in any other vertex of $\mathcal{P}$. (Such an edge exists as at most one edge of G_i is exposed and at most two edges are covered by other vertices of $\mathcal{P}$, and G_i has $\binom{k}{2} \geq 6$ edges.) Let $\mathcal{P}_i = G_i^1, \ldots, G_i^{\ell_i}$ be a path in $\mathcal{T} \setminus \mathcal{P}$ satisfying the following conditions:

a) $E(G_i^1) \cap E(G_i) = \{e_i\}$, and;

b) $\mathrm{dist}_{\mathcal{T}}(G_i^j, G_0) = \mathrm{dist}_{\mathcal{T}}(G_i, G_0) + j$, for all $1 \le j \le \ell_i$, and;

c) The length of the path, ℓ_i, is as large as possible subject to **a)** and **b)**.

To see that a path satisfying **a)** and **b)** exists, let H be any vertex of $\mathcal{T}$ such that $E(H) \cap E(G_i) = \{e_i\}$. Such an H exists, since e_i is not exposed. We have $H \notin V(\mathcal{P})$ since G_i is the only vertex of $\mathcal{P}$ such that $e_i \in E(G_{i'})$. We cannot have that $\mathrm{dist}_{\mathcal{T}}(H, G_0) \le \mathrm{dist}_{\mathcal{T}}(G_i, G_0)$ because then the union of a shortest (G_0, H)-path and the path $G_0, \dots, G_i$ contains an (e_i, f)-path in $\mathcal{T} \setminus \mathcal{P}$ for some $f \in E(\bigcup \mathcal{P}) \setminus \{e_i\}$. We thus have $\mathrm{dist}_{\mathcal{T}}(H, G_0) = \mathrm{dist}_{\mathcal{T}}(G_i, G_0) + 1$ and $H \notin V(\mathcal{P})$ and $E(H) \cap E(G_i) = \{e_i\}$ so that setting $G_i^1 := H$ would give us a path (with one vertex) satisfying **a)** and **b)**.
Let us also observe that, since $\mathrm{dist}_{\mathcal{T}}(G_i, G_0) \ge \mathrm{diam}(\mathcal{T}) - k^{30}$, we must have

$$\ell_i \le k^{30} \text{ for each } i \in I. \tag{2}$$

Also observe that

$$E(G_i^j) \cap E(G_{i'}^{j'}) = \emptyset, \text{ for all } i \ne i' \in I \text{ and } 1 \le j \le \ell_i, 1 \le j' \le \ell_{i'}. \tag{3}$$

(Otherwise there would be a $(e_i, e_{i'})$-path in $\mathcal{H}(G) \setminus \mathcal{P}$, contradicting the choice of I.) Similarly,

$$E(G_i^j) \cap E\left(\bigcup \mathcal{P}\right) = \begin{cases} \{e_i\} & \text{if } j = 1 \\ \emptyset & \text{otherwise,} \end{cases} \tag{4}$$

holds for all $i \in I$ and $1 \le j \le \ell_i$. Let $\mathcal{S}$ denote the subgraph of $\mathcal{T}$ given by

$$\mathcal{S} := \mathcal{P} \cup \bigcup_{i \in I} \mathcal{P}_i.$$

Observe that, using (2),

$$v(\mathcal{S}) \le (k^{30} + 1)^2 < k^{997}. \tag{5}$$

Let us fix an $i \in I$ such that $G_i^{\ell_i}$ is a k-clique. Pick an edge $e \in E(G_i^{\ell_i}) \setminus E(G_i^{\ell_i - 1})$ that is not exposed. Such an edge exists as one edge of $G_i^{\ell_i}$ contained in $G_i^{\ell_i - 1}$ and at most one edge is exposed. Observe that, by **b)**, e is not contained in $E(G_i^j)$ for all $j < \ell_i - 1$. By (3) and (4) the edge e is also not contained in $E(G)$ for any $G \in V(\mathcal{S}) \setminus \{G_i^{\ell_i}\}$. Since e is not exposed, there is a H such that $E(H) \cap E(G_i^{\ell_i}) = \{e\}$. It follows from the previous observations that $H \notin V(\mathcal{S})$. By the maximality of ℓ_i we must have that $\mathrm{dist}_{\mathcal{T}}(H, G_0) \le \mathrm{dist}_{\mathcal{T}}(G_i^{\ell_i}, G_0)$. The union of a shortest (G_0, H)-path and the path $G_0, \dots, G_i, G_i^1, \dots, G_i^{\ell_i}$ therefore contains an (e, f)-path in $\mathcal{H}(G) \setminus \mathcal{S}$ for some edge $f \in E(\bigcup \mathcal{S}) \setminus \{e\}$.
Combining this with (5), properties **(C)** and **(D)** imply

$$|I| \le 8 + 2k^{16} < k^{29}.$$

But this contradicts (1)! It follows that $\mathrm{diam}(\mathcal{T}) \le k^{30}$, and Claim 1 is proved. ■

Claim 2 *At most eight vertices of $\mathcal{T}$ are not k-cliques.*

Proof of Claim 2: If there were nine vertices of $\mathcal{T}$ that are not k-cliques, then there would also be a connected subgraph $\mathcal{S} \subseteq \mathcal{T}$ that contains those nine vertices and $v(\mathcal{T}) \leq 1 + 8\,\mathrm{diam}(\mathcal{T}) \leq 1 + 8k^{30} < k^{997}$. But this would contradict **(C)**. ■

Let us fix a vertex $G_0 \in V(\mathcal{T})$. We will say that a vertex $H \in V(\mathcal{T})$ is *locally maximal* if $\mathrm{dist}_{\mathcal{H}}(H', G_0) \leq \mathrm{dist}_{\mathcal{H}}(H, G_0)$ for all neighbors H' of H.

Claim 3 *$\mathcal{T}$ has at most k^{500} locally maximal vertices.*

Proof of Claim 3: Aiming for a contradiction, suppose that there are at least $k^{500} + 1$ locally maximal vertices. By Claim 2, at least $k^{500} - 7 > k^{499}$ of these locally maximal vertices are k-cliques.

Suppose first there is some edge $e \in E(\bigcup \mathcal{T})$ and $K = k^{16} + 1$ locally maximal vertices $C_1, \ldots, C_K \in V(\mathcal{T})$ that are k-cliques with $e \in E(C_i)$ for all $i = 1, \ldots, K$. By construction of $\mathcal{H}(G)$ this implies that $E(C_i) \cap E(C_j) = \{e\}$ for all $i \neq j$. Let $\mathcal{S} \subseteq \mathcal{T}$ denote the subgraph induced by $C_1, \ldots, C_K$. Observe that

$$\mathrm{dist}_{\mathcal{H}}(C_1, G_0) = \cdots = \mathrm{dist}_{\mathcal{H}}(C_K, G_0),$$

because the C_is are neighbors and locally maximal.

Each C_i can have at most one exposed edge. Let $e_i \neq f_i \in E(C_i) \setminus \{e\}$ be non-exposed edges; and let $H, H' \in V(\mathcal{T})$ be such that $E(C_i) \cap E(H) = \{e_i\}$ and $E(C_i) \cap E(H') = \{f_i\}$. Since $e \notin H$ and $e \notin H'$, we know that H and H' are not in $\mathcal{S}$. As C_i is locally maximal and H, H' are neighbors of C_i, they are at least as close to G_0 as C_i is. This implies that there is a (e_i, f_i)-path in $\mathcal{T} \setminus \mathcal{S}$. (We take a shortest path from H to G_0 and a shortest path from H' to G_0, these cannot contain any vertex of $\mathcal{S}$. Now we leave out some vertices of the union of these two paths if necessary to obtain a path between H and H'.)

Since each pair C_i, C_j meets exactly in e, the K pairs of edges $\{e_1, f_1\}, \ldots, \{e_K, f_K\}$ are all distinct. But this contradicts property **(D)**!

It must be the case that every edge of $E(\bigcup \mathcal{T})$ is contained in at most k^{16} locally maximal vertices that are k-cliques; and using **(C)** at most $k^{16}+8$ vertices including non-k-cliques. We can then find locally maximal $C_1, \ldots, C_L \in V(\mathcal{T})$ that are k-cliques, such that

$$\mathrm{dist}_{\mathcal{H}}(C_i, C_j) \geq 3, \text{ for all } 1 \leq i < j \leq L,$$

and

$$\mathrm{dist}_{\mathcal{H}}(G_0, C_1) = \cdots = \mathrm{dist}_{\mathcal{H}}(G_0, C_L),$$

where

$$L = \left\lceil \frac{k^{499}}{\left(\binom{k}{2} k^{16}\right)^2 \cdot \mathrm{diam}(\mathcal{T})} \right\rceil \geq k^{400}.$$

This follows from the fact that $\mathrm{dist}_{\mathcal{H}}(G_0, C) \leq \mathrm{diam}(\mathcal{T})$ for every clique $C \in V(\mathcal{T})$, Claim 1. Let us write $D := \mathrm{dist}_{\mathcal{H}}(G_0, C_1)$.

For each i, let us fix a shortest (G_0, C_i)-path $\mathcal{P}_i$ and let $\mathcal{S} \subseteq \mathcal{T}$ be the union of these paths. Observe that C_i cannot be adjacent to either C_j or the penultimate vertex of $\mathcal{P}_j$, for all $1 \leq i \neq j \leq L$, because $\mathrm{dist}_{\mathcal{H}}(C_i, C_j) \geq 3$. Moreover, C_i is not adjacent to any other vertex of $\mathcal{P}_j$, because that would imply $\mathrm{dist}_{\mathcal{H}}(C_i, G_0) < D$.

Now pick an $1 \le i \le L$, and let A_i be a neighbor of C_i not on $\mathcal{P}_i$ and let e_i be the edge common to A_i and C_i. (Such a neighbor exists because C_i has at most one exposed edge. Hence there are two edges $e \neq f \in E(C_i)$ and two neighbors H, H' of C_i such that $E(C_i) \cap E(H) = \{e\}, E(C_i) \cap E(H') = \{f\}$. At most one of H, H' can be the penultimate vertex of $\mathcal{P}_i$. Neither H nor H' can be any other vertex of $\mathcal{P}_i$ because $\mathcal{P}_i$ is a shortest path between G_0 and C_i.)

Since $\text{dist}_{\mathcal{H}}(C_j, C_i) \ge 3$, we cannot have that A_i is either C_j or the penultimate vertex of $\mathcal{P}_j$ (for all $j \neq i$). We also cannot have that A_i is any other vertex of $\mathcal{P}_j$ because that would imply $\text{dist}_{\mathcal{H}}(C_i, G_0) < D$. We see that A_i does in fact not belong to $\mathcal{S}$ altogether. Since C_i is locally maximal we have $\text{dist}_{\mathcal{H}}(A_i, G_0) \in \{\text{dist}_{\mathcal{H}}(C_i, G_0) - 1, \text{dist}_{\mathcal{H}}(C_i, G_0)\}$. The union of a shortest (G_0, A_i)-path and the path $\mathcal{P}_i$ therefore contains an (e_i, f_i)-path in $\mathcal{H}(G) \setminus \mathcal{S}$ for some $f_i \in E(\bigcup \mathcal{S}) \setminus \{e_i\}$.

Observe that the pairs $\{e_i, f_i\}$ are distinct (the e_is are distinct, as $E(C_i) \cap E(C_j) = \emptyset$ for all $j \neq i$). Thus $\mathcal{S}$ has $L > k^{16}$ pairs of edges $e \neq f \in E(\bigcup \mathcal{S})$ such that there is an (e, f)-path in $\mathcal{H}(G) \setminus \mathcal{S}$. Also observe that $\mathcal{S} \subseteq \mathcal{H}(G)$ is a connected subgraph with

$$v(\mathcal{S}) \le \text{diam}(\mathcal{T}) \cdot L \le k^{30} \cdot k^{499} < k^{997}.$$

But this contradicts property **(D)**! It follows that there are at most k^{500} locally maximal vertices in $\mathcal{T}$ as claimed. ∎

Let us observe that every vertex of $\mathcal{T}$ lies on a shortest path between G_0 and some locally maximal G'. Hence, combining Claims 1 and 3 with property **(E)**, we see that

$$v(\mathcal{T}) \le k^{500} \cdot 9 \cdot \text{diam}(\mathcal{T}) \le k^{532} < k^{997}.$$

By property **(B)**, we then get that $v(\bigcup \mathcal{T}) \le 9k \cdot k^{997} \le k^{1000}$. We now see, using property **(A)**, that $\bigcup \mathcal{T}$ has maximum density at most $\frac{k+1}{2}$. By Lemma 2.1, Breaker can therefore win on $\bigcup \mathcal{T}$, which concludes the proof of Lemma 2.2. ∎

Our next goal will be to show that $G(n, p)$ satisfies properties **(A)**-**(E)** a.a.s. for a suitable choice of p.

Since there are only finitely many isomorphism classes of graphs on at most k^{1000} vertices, Theorem 1.5 immediately implies the following statement.

Corollary 2.3 *If $p = p(n) = O(n^{-\frac{2}{k+1}})$, then $G(n,p)$ satisfies property* **(A)** *a.a.s.*

Next, for a graph G, we define

$$\varphi(G) := e(G) - v(G) \cdot \frac{k+1}{2}.$$

Note that the density of G is greater than $\frac{k+1}{2}$ if and only if $\varphi(G) > 0$.

We will make heavy use the following two simple facts.

Lemma 2.4 *Let G_1 and G_2 be two graphs. We have*

$$\begin{aligned}\varphi(G_1 \cup G_2) = &\varphi(G_1) + \varphi(G_2) - |E(G_1) \cap E(G_2)| + \\ &+ |V(G_1) \cap V(G_2)| \cdot \frac{k+1}{2} \qquad (6)\\ \geq &\varphi(G_1) + \varphi(G_2) - \binom{|V(G_1) \cap V(G_2)|}{2} + \\ &+ |V(G_1) \cap V(G_2)| \cdot \frac{k+1}{2}. \qquad (7)\end{aligned}$$

The proof of this lemma follows directly from the definition of φ.

If in an application of the merging rule we have that $V(G_2)$ and $V(G_1)$ are incomparable (i.e. neither is a subset of the other), then we call the merge *proper*. Otherwise, we speak of an *improper* merge.

If in a merge we have $|V(G_1) \cap V(G_2)| = s$, then we speak of an s-merge, and if $|V(G_1) \cap V(G_2)| \geq s$ we speak of an $(\geq s)$-merge.

Lemma 2.5 *Let G be a graph and let $\mathcal{F} = \{C_1, \dots, C_N\}$ be a family of k-cliques such that $G = \bigcup \mathcal{F}$ and G can be obtained from $\mathcal{F}$ via a sequence of merges. Then there is also such a sequence of merges such that every merge is either:*

(i) *an improper merge, or;*

(ii) *an s-merge with $3 \leq s \leq k-1$, or;*

(iii) *a k-merge with $|E(G_1) \cap E(G_2)| \leq 2$.*

Proof. Suppose that the statement is false, and let G, $\mathcal{F} = \{C_1, \dots, C_N\}$ be a counterexample with $e(G)$ as small as possible.

Let us say a merge is *desired* if it is of one of the types **(i)**-**(iii)** in the statement of the lemma above. Consider a sequence of merges, and suppose that until some point in the sequence we have only done desired merges to arrive at a collection of graphs $\mathcal{F}' = \{G_1, \dots, G_M\}$ and now there is a merge possible, but no desired merge is possible. Say we can do a merge on G_i, G_j.

We can partition $\mathcal{F}$ into collections $\mathcal{F}_1, \dots, \mathcal{F}_M$ such that G_i can be obtained from $\mathcal{F}_i$ by desired merges. If $G_i \cup G_j \neq G$ then, by choice (minimality) of G, there is a sequence of desired merges that produces $G_i \cup G_j$ from $\mathcal{F}_i \cup \mathcal{F}_j$. This also gives a sequence of desired merges that produces $\{G_i \cup G_j\} \cup (\mathcal{F}' \setminus \{G_i, G_j\})$ from $\mathcal{F}$. Hence, repeating this argument, we can conclude that we must have $M = 2$ and $\mathcal{F}' = \{G_1, G_2\}$ with $G_1 \cup G_2 = G$ and $V(G_1) \cap V(G_2) \geq k$. We can assume the labeling is such that $e(G_1) \geq e(G_2)$. We now claim that there is a sequence of desired merges that produces a collection of graphs $\{G'_1, \dots, G'_K\}$ from $\mathcal{F}$ with $e(G'_1) > e(G_1)$. Observe that it suffices to prove the claim as repeated applications of it will prove the lemma.

To prove the claim, first observe that there is also a sequence of desired merges that produces $\{G_1\} \cup \mathcal{F}_2$. If there is a $C \in \mathcal{F}_2$ with $3 \leq |V(G_1) \cap V(C)| \leq k-1$, then we are done, so we assume this is not the case. Similarly we can assume there is no $C \in \mathcal{F}_2$ with $V(C) \subseteq V(G_1)$ and $E(C) \not\subseteq E(G_1)$. We extend the sequence of merges as follows. We first merge all $C \in \mathcal{F}_2$ with $V(C) \subseteq V(G_1)$ with G_1, to arrive at a collection $\{G_1, C_1, \dots, C_K\}$ with $|V(G_1) \cap V(C_i)| \leq 2$ for each $1 \leq i \leq K$. We now continue merging, giving priority to merges with G_1. (Observe that as long as we did not yet produce G there is always at least one merge possible, as there is a sequence of merges, possibly not all desired, that produces G from $G_1, C_1, \ldots, C_K$). Suppose that at some point we merge G_1 with some H. If $3 \leq |V(G_1) \cap V(H)| \leq k-1$ we are done (by minimality of G we can also produce H from $C_1, \ldots, C_K$ by only desired merges), so we must have $|V(G_1) \cap V(H)| \geq k$. Since we give priority to merges with G_1, in some earlier merge we must have merged H_1, H_2 to get H and $|V(G_1) \cap V(H_1)|, |V(G_1) \cap V(H_2)| \leq 2$. But then $|E(G_1) \cap E(H_1)| \leq 1$ and $|E(G_1) \cap E(H_2)| \leq 1$, so the merge on G_1, H is of type **(iii)**. (As an aside, note that $k \leq |V(G_1) \cap V(H)| \leq |V(G_1) \cap V(H_1)| + |V(G_1) \cap V(H_2)| \leq 4$ so we must have $|V(G_1) \cap V(H)| = k = 4$.) Thus we can produce $G_1' := G_1 \cup H$ from $C_1, \ldots, C_N$ by only desired merges. Observe that

$$\begin{aligned} e(G_1') &= e(H) + e(G_1) - e(H \cap G_1) \\ &\geq e(G_1) + \tbinom{k}{2} - \max(2, \tbinom{k-1}{2}) \\ &> e(G_1) \end{aligned}$$

This proves the claim, and concludes the proof of the lemma. ■

Lemma 2.6 *Let G be a graph and let $\mathcal{F} = \{C_1, \ldots, C_N\}$ be a family of k-cliques such that $G = \bigcup \mathcal{F}$ and G can be obtained from $\mathcal{F}$ via a sequence of merges. Let a denote the least number of* proper *merges that occurs in any sequence of merges as provided by Lemma 2.5. Then*

$$\varphi(G) \geq \frac{(a-2)(k-3)-6}{2}. \tag{8}$$

Proof. Let t denote the number of merges (proper and improper) in the sequence that Lemma 2.5 provides. The proof is by induction on t. If $t = a = 0$, then $G = K_k$ and we can easily see from the definition that

$$\varphi(K_k) = \binom{k}{2} - k\frac{k+1}{2} = -k,$$

and this fits the formula (8).

Next, suppose that the statement holds for all $t' < t$. In the last step of the merge sequence we merged two graphs G_1, G_2 to obtain G. Let a_1 denote the number of proper merges that occur in the merge sequence for G_1 and a_2 in the merge sequence for G_2.

First suppose the merge on G_1, G_2 is improper, say $V(G_2) \subseteq V(G_1)$. Observe that in this case, there is also a merge sequence for G with only a_1 proper merges (first we create G_1, and then it absorbs each of the cliques that went into G_2). Hence $a = a_1$, and by definition of φ and the induction hypothesis, we have

$$\varphi(G) \geq \varphi(G_1) = \frac{(a-2)(k-3)-6}{2}.$$

Now assume that $3 \leq s := |V(G_1) \cap V(G_2)| \leq k-1$. Then, applying (7) from Lemma 2.4, we have

$$\begin{aligned} \varphi(G) &\geq \varphi(G_1) + \varphi(G_2) - \binom{s}{2} + s\frac{k+1}{2} \\ &= \varphi(G_1) + \varphi(G_2) + s\frac{k-s+2}{2} \\ &\geq \varphi(G_1) + \varphi(G_2) + 3\frac{k-1}{2}. \end{aligned}$$

Similarly, if $s \geq k$ and $|E(G_1) \cap E(G_2)| \leq 2$, then by (6) we get

$$\begin{aligned}\varphi(G) &\geq \varphi(G_1) + \varphi(G_2) - 2 + k\frac{k+1}{2} \\ &= \varphi(G_1) + \varphi(G_2) + \frac{k^2+k-4}{2} \\ &\geq \varphi(G_1) + \varphi(G_2) + 3\frac{k-1}{2}.\end{aligned}$$

Hence, in both cases we have

$$\begin{aligned}\varphi(G) &\geq \frac{(a_1-2)(k-3)-6}{2} + \frac{(a_1-2)(k-3)-6}{2} + 3\frac{k-1}{2} \\ &= \frac{(a-2)(k-3)-6}{2},\end{aligned}$$

using $a = a_1 + a_2 + 1$, and the induction hypothesis. This shows that the statement of the lemma holds. ■

Lemma 2.7 *Let G be a graph that satisfies property* **(A)**. *Then G also satisfies property* **(B)**.

Proof. For a contradiction, suppose that there exists $H \in \mathcal{F}(G)$ with $v(H) > 9k$. Let us first observe that there also exists a $H' \subseteq H$ with $9k < v(H') \leq k^{1000}$ that can be obtained from a subset of $\mathcal{F}^{(0)}$ via a sequence of merges. (To see this, suppose $v(H') > k^{1000}$ and undo the last merge we did to create H. The largest of the two graphs has $> 9k$ vertices. If it has more than k^{1000} vertices then we break it into two again, and repeat.) Consider the number a for H' from Lemma 2.6. We have $\varphi(H') = ((a-2)(k-3)-6)/2$. It cannot be that $a \geq 9$, because then H' would have density strictly greater than $\frac{k+1}{2}$. Hence $a \leq 8$, so that $v(G) \leq (k-3)a + k \leq 9k$, a contradiction. ■

Combining Lemma 2.7 with Corollary 2.3 we immediately have the following.

Corollary 2.8 *If $p = p(n) = O(n^{-\frac{2}{k+1}})$, then $G(n,p)$ satisfies property* **(B)** *a.a.s.*

Lemma 2.9 *Let G be an arbitrary graph, and let $\mathcal{T}, \mathcal{T}' \subseteq \mathcal{H}(G)$ be connected subgraphs of $\mathcal{H}(G)$ with $\mathcal{T} \subseteq \mathcal{T}'$. Then $\varphi(\bigcup \mathcal{T}) \leq \varphi(\bigcup \mathcal{T}')$.*

Proof. By definition of $\mathcal{H}$, there is a sequence of k-cliques $C_1, \ldots, C_N$ such that $\bigcup \mathcal{T} = \bigcup_{i=1}^{N_0} C_i$ and $\bigcup \mathcal{T}' = \bigcup_{i=1}^{N} C_i$ with $N_0 \leq N$. (We just break every non-clique vertex of $\mathcal{T}'$ up into its constituent cliques, and then choose the indices such that the cliques belonging to $\mathcal{T}'$ come first.) Moreover, since $\mathcal{T}, \mathcal{T}'$ are both connected, we can assume that the ordering is such that $\left|V(C_i) \cap \left(\bigcup_{j=1}^{i-1} C_j\right)\right| \geq 2$ for all $i \geq 2$.

Let us pick an arbitrary $2 \leq i \leq N$. Since $2 \leq s := \left|V(C_i) \cap \left(\bigcup_{j=1}^{i-1} V(C_j)\right)\right| \leq k$, by (6) and recalling that $\varphi(K_k) = -k$, we see that

$$\begin{aligned}\varphi\left(\cup_{j=1}^{i} C_j\right) &\geq \varphi\left(\cup_{j=1}^{i-1} C_j\right) - k - \binom{s}{2} + s\frac{k+1}{2} \\ &= \varphi\left(\cup_{j=1}^{i-1} C_j\right) + \frac{(s-2)(k-s)}{2} \\ &\geq \varphi\left(\cup_{j=1}^{i-1} C_j\right).\end{aligned}$$

The lemma now follows from repeated applications of this inequality. ■

Lemma 2.10 *Let G be an arbitrary graph. If $G_1, \ldots, G_N \in \mathcal{F}(G)$ induce a connected subgraph of $\mathcal{H}(G)$ and $G_1, \ldots, G_9$ are not k-cliques, then $\varphi(\bigcup_{i=1}^N G_i) \geq \frac{1}{2}$;*

Proof. Let $\mathcal{T}$ denote the subgraph of $\mathcal{H}(G)$ induced by $G_1, \ldots, G_N$.

We claim that it is enough to look at the case when $G_1, \ldots, G_9$ all have degree at most 8 in $\mathcal{T}$. To see this, observe that we can construct an induced subgraph $\mathcal{T}'$ of $\mathcal{T}$ by starting with a shortest path between G_1 and G_2, and repeatedly add a shortest path between the current graph and the nearest G_j $(3 \leq j \leq 9)$ that is not yet incident with any vertex of the current subgraph. This way each G_i will have degree at most eight. Now, if we prove the statement for $\mathcal{T}'$, Lemma 2.9 readily implies it for $\mathcal{T}$.

Moreover, we can assume that all other vertices of $\mathcal{T}$ are k-cliques. (In the last step of the construction above we added an induced path to a non-k-clique. If the path contains more than one vertex that is not a k-clique we replace it with a shorter path. This way we have constructed an induced subgraph of $\mathcal{T}$ with one fewer non-k-clique vertex, still having at least 9 non-k-cliques. Applying Lemma 2.9, we can repeat as many times as necessary.)

So, $\mathcal{T}$ has exactly 9 vertices that are not k-cliques, and each of them has degree at most 8 in $\mathcal{T}$. Since $\mathcal{T}$ is connected, we can relabel $G_1, \ldots, G_N$ in such a way that G_i is adjacent to one of $G_1, \ldots, G_{i-1}$, and G_1 is not a k-clique. By Lemma 2.6 above (applied with $a = 2$) we have

$$\varphi(G_1) \geq -\frac{k+3}{2}. \tag{9}$$

Now notice that, whenever G_i is a k-clique then, writing $s := \left|V(G_i) \cap \left(\bigcup_{j=1}^{i-1} V(G_j)\right)\right|$, similarly as before we have

$$\begin{aligned} \varphi\left(\cup_{j=1}^{i} G_j\right) &\geq \varphi\left(\cup_{j=1}^{i-1} G_j\right) - k - \binom{s}{2} + s\frac{k+1}{2} \\ &= \varphi\left(\cup_{j=1}^{i-1} G_j\right) + \frac{(s-2)(k-s)}{2} \\ &\geq \varphi\left(\cup_{j=1}^{i-1} G_j\right), \end{aligned} \tag{10}$$

using that $\varphi(G_i) = -k$ and $2 \leq s \leq k$ as G_i is a k-clique.

Now suppose that G_i is not a k-clique. We have $\varphi(G_i) \geq -(k+3)/2$. Since the degree of G_i in $\mathcal{T}$ is at most eight, and $\mathcal{T}$ is an induced subgraph of $\mathcal{H}(G)$, we know that $t := \left|E(G_i) \cap \left(\bigcup_{j=1}^{i-1} E(G_j)\right)\right| \leq 8$. Let us write $s := \left|V(G_i) \cap \left(\bigcup_{j=1}^{i-1} V(G_j)\right)\right|$ as before. Notice that, if $t \geq 7$, then $s \geq 5$ (this is because $\binom{4}{2} = 6 < 7$); if $4 \leq t \leq 6$, then $s \geq 4$; if $2 \leq t \leq 3$, then $s \geq 3$; and if $t = 1$, then $s \geq 2$. We therefore have, using (9), that

$$\begin{aligned} \varphi(\bigcup_{j=1}^{i} G_j) &\geq \varphi(\bigcup_{j=1}^{i-1} G_j) - \tfrac{k+3}{2} - t + s\tfrac{k+1}{2} \\ &\geq \varphi(\bigcup_{j=1}^{i-1} G_j) - \tfrac{k+3}{2} + \\ &\quad + \min\left(-1 + 2\tfrac{k+1}{2}, -3 + 3\tfrac{k+1}{2}, -6 + 4\tfrac{k+1}{2}, -8 + 5\tfrac{k+1}{2}\right) \\ &= \varphi(\bigcup_{j=1}^{i-1} G_j) - \tfrac{k+3}{2} + k \\ &= \varphi(\bigcup_{j=1}^{i-1} G_j) + \tfrac{k-3}{2}. \end{aligned} \tag{11}$$

Thus, combining (9), (10) and (11) we have

$$\varphi\left(\cup_{i=1}^{N} G_i\right) \geq -\frac{k+3}{2} + 8 \cdot \frac{k-3}{2} = \frac{7k-27}{2} \geq \frac{1}{2},$$

as required. ■

Observe that if property **(B)** holds, then for every $\mathcal{T} \subseteq \mathcal{H}(G)$ we have $v(\bigcup \mathcal{T}) \leq 9k \cdot v(\mathcal{T}) \leq k^3 \cdot v(\mathcal{T})$. If in addition property **(A)** holds, then, by Lemma 2.10, no connected subgraph $\mathcal{T} \subseteq \mathcal{H}(G)$ with $v(\mathcal{T}) \leq k^{997}$ can contain more that eight vertices that are not k-cliques. Hence, we proved the next statement.

Corollary 2.11 *If a graph G satisfies properties* **(A)** *and* **(B)***, then it also satisfies property* **(C)***.*

Thus, by Corollaries 2.3 and 2.8 we also have the following.

Corollary 2.12 *If $p = p(n) = O\left(n^{-\frac{2}{k+1}}\right)$, then $G(n,p)$ satisfies* **(C)** *a.a.s.*

With the next statement we continue preparing our ground for dealing with property **(D)**.

Lemma 2.13 *Let G be an arbitrary graph, and $k \geq 4$. If $G_1, \dots, G_N \in V(\mathcal{H}(G))$ are distinct and $s_i := |V(G_i) \cap V(\bigcup_{j<i} G_j)| \geq 2$ for all $1 < i \leq N$ and there are at least nine indices i for which $s_i \geq 3$, then $\varphi(\bigcup_{j=1}^N G_j) \geq 1/2$.*

Proof. Let us set $t_i := |E(G_i) \cap E\left(\cup_{j<i} G_j\right)|$. Observe that, from the definition of $\mathcal{H}$, we have $|E(G_i) \cap E(G_j)| \leq 1$ for all $i \neq j$.

We now repeat the following action until it is no longer possible to apply it.

- Action: If there is some $1 < i < N$ such that $t_{i+1} > 2$ and $t_{i+1} - t_i > 1$, then we swap G_i and G_{i+1}.

Consider a single application of the action. Since before the application we have $s_i^{\text{old}} \geq 2$ and $t_{i+1}^{\text{old}} > 2$ then, after we apply the rule, we have $s_{i+1}^{\text{new}} \geq s_i^{\text{old}} \geq 2$ and $t_i^{\text{new}} \geq t_{i+1}^{\text{old}} - 1 \geq 2$, and hence $s_i^{\text{new}} \geq 3$. In particular, note that if $s_i^{\text{old}} \geq 3$, then also $s_{i+1}^{\text{new}} \geq 3$ and if $s_{i+1}^{\text{old}} \geq 3$, then $s_i^{\text{new}} \geq 3$, so the number of indices i with $s_i \geq 3$ does not decrease. All the other values of s_j and t_j for $j \notin \{i, i+1\}$ obviously remain the same.

To see that the action can only be applied finitely many times, first note that after each application we have either $t_i^{\text{new}} = t_{i+1}^{\text{old}}$ and $t_{i+1}^{\text{new}} = t_i^{\text{old}}$, or $t_i^{\text{new}} = t_{i+1}^{\text{old}} - 1$ and $t_{i+1}^{\text{new}} = t_i^{\text{old}} + 1$. So, in both cases $t_i^{\text{new}} + t_{i+1}^{\text{new}} = t_i^{\text{old}} + t_{i+1}^{\text{old}}$. If we define $F(t_1, \dots, t_N) := \sum_{i=1}^N i \cdot t_i$, after each action we have

$$
\begin{aligned}
F(t_1^{\text{old}}, \dots, t_N^{\text{old}}) - F(t_1^{\text{new}}, \dots, t_N^{\text{new}}) &= i \cdot t_i^{\text{old}} + (i+1) \cdot t_{i+1}^{\text{old}} - i \cdot t_i^{\text{new}} - (i+1) \cdot t_{i+1}^{\text{new}} \\
&= i\left(t_i^{\text{old}} + t_{i+1}^{\text{old}} - t_i^{\text{new}} - t_{i+1}^{\text{new}}\right) + t_{i+1}^{\text{old}} - t_{i+1}^{\text{new}} \\
&= t_{i+1}^{\text{old}} - t_{i+1}^{\text{new}} \\
&\geq t_{i+1}^{\text{old}} - (t_i^{\text{old}} + 1) \\
&> 0.
\end{aligned}
$$

Hence, F decreases after every action. Since F is always a nonnegative integer, there can thus be only finitely many actions.

To sum up, once we are no longer able to apply this action, in the resulting order $G_1, \dots, G_N$ we will still have $s_i \geq 2$ for all $1 < i \leq N$; and there are still at least nine indices i such that $s_i \geq 3$. Observe that in addition we have that if $t_{i+1} > 2$ for some i, then $t_{i-1} - t_i \leq 1$.

Let $i_1 < \cdots < i_9$ be indices such that $s_{i_j} \geq 3$ for $1 \leq j \leq 9$. Let us first assume that $t_{i_j} < 5$ for all $1 \leq j \leq 9$. In this case we have, for all $j > 1$,

$$\begin{aligned}
\varphi\left(\bigcup_{\ell=1}^{i_j} G_\ell\right) &= \varphi\left(\bigcup_{\ell=1}^{i_j-1} G_\ell\right) + \varphi(G_{i_j}) - t_{i_j} + s_{i_j}\tfrac{k+1}{2} \\
&\geq \varphi\left(\bigcup_{\ell=1}^{i_j-1} G_\ell\right) + \varphi(G_{i_j}) + \\
&\quad + \min\left(3\tfrac{k+1}{2}, -1 + 3\tfrac{k+1}{2}, -2 + 3\tfrac{k+1}{2},\right. \\
&\qquad \left. -3 + 3\tfrac{k+1}{2}, -4 + 4\tfrac{k+1}{2}\right) \\
&\geq \varphi\left(\bigcup_{\ell=1}^{i_j-1} G_\ell\right) + \tfrac{k-3}{2} \\
&\geq \varphi\left(\bigcup_{\ell=1}^{i_{j-1}} G_\ell\right) + \tfrac{k-3}{2}.
\end{aligned} \tag{12}$$

(Here we used that four edges span at least four vertices in the second line; $\varphi(G_{i_j}) \geq -k$ by Lemma 2.6 for the third line; and Lemma 2.9 for the last line. Notice the difference between $i_j - 1$ and i_{j-1} in the last two lines.) Observe that, again by Lemma 2.9, we have

$$\varphi(\bigcup_{\ell=1}^{i_1-1} G_\ell) \geq \varphi(G_1) \geq -k. \tag{13}$$

Combining (12) and (13) with Lemma 2.9 we see that

$$\begin{aligned}
\varphi\left(\bigcup \mathcal{T}\right) &\geq \varphi(\bigcup_{j=1}^{i_9} G_j) \\
&\geq -k + 9\tfrac{k-3}{2} \\
&= \tfrac{7k-27}{2} \\
&\geq \tfrac{1}{2}.
\end{aligned}$$

It remains to consider the case when $t_{i_j} \geq 5$ for some $1 \leq j \leq 9$. In this case there are also indices $j_1 < j_2 < j_3 < j_4$ such that $t_{j_\ell} = \ell + 1$. Hence we find (again using Lemma 2.9) that

$$\begin{aligned}
\varphi(\bigcup_{i=1}^{N} G_i) &\geq \varphi(G_1) + \left(-k - 2 + 3\tfrac{k+1}{2}\right) + \left(-k - 3 + 3\tfrac{k+1}{2}\right) \\
&\quad + \left(-k - 4 + 4\tfrac{k+1}{2}\right) + \left(-k - 5 + 4\tfrac{k+1}{2}\right) \\
&\geq -5k - 14 + 14\tfrac{k+1}{2} \\
&= \tfrac{4k-14}{2} \\
&\geq 1/2,
\end{aligned}$$

as required. ■

Lemma 2.14 *Let $k \geq 4$. Then there is $c = c(k)$ such that if $p = p(n) \leq cn^{-\frac{2}{k+1}}$, then $G(n,p)$ satisfies property* **(D)** *a.a.s.*

Proof. By Corollaries 2.3 and 2.8 we know that $G(n,p)$ satisfies properties **(A)** and **(B)** a.a.s., if $p = O\left(n^{-\frac{2}{k+1}}\right)$.

A rough outline of the proof we are about to give is as follows. We shall now first describe a structure that every graph G that satisfies properties **(A)**, **(B)** but not **(D)** must contain; and then we will show that $G(n,p)$ with $p = cn^{-\frac{2}{k+1}}$ does not contain such a structure a.a.s. (if c is chosen appropriately small) by means of a first moment calculation.

Thus, let G be an arbitrary graph that satisfies **(A)** and **(B)** but does not satisfy **(D)**. Observe that for $\mathcal{T} \subseteq \mathcal{H}(G)$ a connected subgraph of $\mathcal{H}(G)$, each path in $\mathcal{H}(G) \setminus \mathcal{T}$ is an (e,f)-path for at most $\binom{9k}{2}^2$ pairs $e \neq f \in E(\bigcup \mathcal{T})$. Suppose $v(\mathcal{T}) \leq k^{997}$, and there are at least $k^{16}+1$ pairs of edges $e \neq f \in E(\bigcup \mathcal{T})$ for which there is an (e,f)-path in $\mathcal{H}(G) \setminus \mathcal{T}$.

We can then construct an increasing sequence of subgraphs $\mathcal{T}^{(0)} = \mathcal{T} \subseteq \mathcal{T}^{(1)} \subseteq \cdots \subseteq \mathcal{T}^{(9)}$ of $\mathcal{H}(G)$ as follows, such that $\mathcal{T}^{(i)}$ is obtained from $\mathcal{T}^{(i-1)}$ by adding a path $\mathcal{P}^{(i)} = G_1^i, \ldots, G_{\ell_1}^i$ from $\mathcal{H}(G)$ with the following properties

- $V(G_j^i) \cap V\left(\bigcup \mathcal{T}^{(i-1)}\right) = \emptyset$, for all $1 < j < \ell_i$, and;
- $\left|V(G_j^i) \cap V\left(\bigcup \mathcal{T}^{(i-1)} \cup \bigcup_{j'<j} G_{j'}^i\right)\right| = 2$ for all $j < \ell_i$, and;
- $\left|V(G_{\ell_i}^i) \cap V\left(\bigcup \mathcal{T}^{(i-1)} \cup \bigcup_{j<\ell_i} G_j^i\right)\right| \geq 3$.

To see that such a path exists, observe that there is a pair of edges $e \neq f \in E(\bigcup \mathcal{T})$ that is not contained in the endpoints of previous paths (i.e. at least one of e_i, f_i is not contained in $\bigcup_{\substack{i'<i,\\ j\leq \ell_{i'}}} E(G_j^{i'})$) such that there is an (e,f)-path in $\mathcal{H} \setminus \mathcal{T}$. To support this claim, observe that the endpoints of $\mathcal{P}^{(1)}, \ldots, \mathcal{P}^{(i-1)}$ cover at most $2(i-1)\binom{9k}{2}$ edges of $E(\bigcup \mathcal{T})$ and hence there are at most $4(i-1)^2\binom{9k}{2}^2 \leq 256\binom{9k}{2}^2 < k^{16}+1$ pairs of edges covered by these previous endpoints. We now look at that (e,f)-path, and substitute it with its subpath if needed, to ensure the above demands are met.

Observe from **(B)** that for a connected subgraph our construction yields an ordering of $V(\mathcal{T}^{(9)})$ where each $H \in V(\mathcal{T}^{(9)})$, except the first in the ordering, intersects the union of the graphs that come before it in at least two vertices, and $G_{\ell_i}^i$ intersects the union of the graphs that are before it in the sequence in at least three vertices for each $i = 1, \ldots, 9$. By Lemma 2.13 above, we have

$$\varphi\left(\bigcup \mathcal{T}^{(9)}\right) \geq 1/2. \tag{14}$$

Also observe that, given $\ell_1, \ldots, \ell_9 \geq 1$, the number of edges of $\bigcup \mathcal{T}^{(9)}$ is at least

$$e\left(\bigcup \mathcal{T}^{(9)}\right) \geq (\ell - 9)\left(\binom{k}{2} - 1\right) \geq 5(\ell - 9), \tag{15}$$

where $\ell := \ell_1 + \cdots + \ell_9$.

We will bound the expectation of the number of subgraphs $\mathcal{T}^{(9)}$ of $\mathcal{H}(G(n,p))$ that can be constructed in this way starting from some connected $\mathcal{T} \subseteq \mathcal{H}(G(n,p))$ with at most k^{997} vertices.

For convenience let us write $v_0 := 9k \cdot k^{997}$, and let $C(m)$ denote the number of (labeled) connected graphs on at most m vertices. For $\ell_1, \ldots, \ell_9 \geq 1$, let us write

$$\mathcal{I}(\ell_1, \ldots, \ell_9) \quad := \quad C(v_0) \cdot C^{\ell}(9k) \cdot \binom{9k}{2}^{2(\ell-9)} \cdot \left(\sum_{j=2}^{9k} \binom{v_0+9k\ell}{j}\binom{9k}{j}\right)^{18}.$$

Observe that $\mathcal{I}(\ell_1, \ldots, \ell_9)$ is an upper bound on the number of isomorphism classes of graphs that can arise as $\bigcup \mathcal{T}^{(9)}$, if we fix the path lengths $\ell_1, \ldots, \ell_9$. To see this, note that $C(v_0)$ is an upper bound on the number of isomorphism classes that $\mathcal{T}^{(0)}$ can have; $C(9k)$ is an upper bound on the number of isomorphism classes of G^i_j; $\binom{9k}{2}^2$ is an upper bound on the number of ways we can identify an edge of G^i_{j-1} with an edge of G^i_j; and $\left(\sum_{j=2}^{9k}\binom{v_0+9k\ell}{j}\binom{9k}{j}\right)$ is an upper bound on the number of ways we can identify a subset of at least two of the vertices of G^i_1, or of $G^i_{\ell_i}$, with vertices of the rest of the graph.

Observe that, writing $\alpha := C(9k)\cdot\binom{9k}{2}^2$, we have

$$\mathcal{I}(\ell_1,\dots,\ell_9) = O\left(\ell^{162k}\alpha^\ell\right).$$

Now let X denote the number of subgraphs of $H \subseteq G(n,p)$ that are of the form $\bigcup \mathcal{T}^{(9)}$. Observe that, by (14) and (15) above, if $p = cn^{-\frac{2}{k+1}}$ with $c < 1$ then we have

$$n^{v(H)}p^{e(H)} \le n^{-\frac{1}{k+1}}\cdot c^{e(H)} \le n^{-\frac{1}{k+1}}c^{5(\ell-9)}.$$

Hence, if $p \le cn^{-\frac{2}{k+1}}$, then we can write

$$\begin{aligned}
\mathbb{E}(X) &\le \textstyle\sum_{\ell_1=1}^\infty\cdots\sum_{\ell_9=1}^\infty \mathcal{I}(\ell_1,\dots,\ell_9)\cdot n^{-\frac{1}{k+1}}\cdot c^{5(\ell-9)}\\
&= O\left(n^{-\frac{1}{k+1}}\textstyle\sum_{\ell_1=1}^\infty\cdots\sum_{\ell_9=1}^\infty \ell^{162k}(c^5\alpha)^\ell\right)\\
&= O\left(n^{-\frac{1}{k+1}}\textstyle\sum_{\ell_1=1}^\infty\cdots\sum_{\ell_9=1}^\infty (\ell_1\dots\ell_9)^{162k}(c^5\alpha)^\ell\right)\\
&= O\left(n^{-\frac{1}{k+1}}\left(\textstyle\sum_{x=1}^\infty x^{162k}(c^5\alpha)^x\right)^9\right)\\
&= o(1),
\end{aligned}$$

where the last line holds provided $c < 1/\sqrt[5]{\alpha} = 1/\sqrt[5]{C(9k)\cdot\binom{9k}{2}^2}$.

Hence $G(n,p)$ indeed satisfies property **(D)** a.a.s. if $p \le cn^{-\frac{2}{k+1}}$ for a suitable choice of the constant c. ■

Lemma 2.15 *If a graph G satisfies property* **(A)**, *then it also satisfies property* **(E)**.

Proof. Let pick $H_1, H_2 \in \mathcal{H}(G)$ with $D := \mathrm{dist}_{\mathcal{H}}(H_1, H_2) \le k^{995}$; and let $\mathcal{T}$ be as defined in the statement of property **(E)**. We will construct an ordering of (a subset of) $V(\mathcal{T})$ as in Lemma 2.13 above. We start with $G_0 = H_1, G_1, \dots, G_D = H_2$ an arbitrary (H_1, H_2)-path of length D. We now repeat the following rule until either this is no longer possible or until we have applied it nine times

- Rule: If there is a $H \in \mathcal{T}$ that is not yet contained in the current sequence, then we take a shortest (H_1, H_2)-path containing H and we append it to the sequence, leaving out elements that already occur in the sequence.

Suppose that we were able to apply the rule nine times. Then we have ended up with a sequence $G_1, \dots, G_N$ with $N \le 10D$ that satisfies the conditions of Lemma 2.13. Hence $\varphi(\bigcup_{i=1}^N G_i) \ge 1/2$. On the other hand, by Lemma 2.7, we have $v(\bigcup_{i=1}^N G_i) \le 90kD < k^{997}$. But then property **(A)** implies that $\varphi(\bigcup_{i=1}^N G_i) \le 0$, a contradiction.

It follows that we were not able to iterate the rule more than eight times. Consequently $v(\mathcal{T}) \le 9D$, as required. ■

Combining Corollary 2.3 with Lemma 2.15, we get the following statement.

Corollary 2.16 *If $p = p(n) = O\left(n^{-\frac{2}{k+1}}\right)$, then $G(n,p)$ satisfies property* **(E)** *a.a.s.*

We have now done all the work necessary to prove Theorem 1.1.

Proof of Theorem 1.1: Lemma 2.2 together with Corollaries 2.3, 2.8, 2.12, Lemma 2.14 and Corollary 2.16 immediately imply Theorem 1.1. ■

3 Hitting time for the triangle game

Proof of Theorem 1.3: It is easy to check that Maker can claim a triangle playing as the first player on the graph $K_5 - e$, so he can also win on any graph containing it. Hence, for every graph process G we have $\tau(G; \mathcal{M}_{\mathcal{K}_3}) \leq \tau(G; \mathcal{G}_{\mathcal{K}_5^-})$.

Let $\tilde{G} = \{G_i\}_{i=0}^{\binom{n}{2}}$ be a random graph process. We define $t_1 := \tau(\tilde{G}; \mathcal{M}_{\mathcal{K}_3})$ and $t_2 := \tau(\tilde{G}; \mathcal{G}_{\mathcal{K}_5^-})$, noting that $t_1 \leq t_2$. Since $m(K_5 - e) = 9/5$, we have that $t_2 \leq n^{13/9} \log n$ a.a.s. (see [14]). From now on we will assume that this holds.

Let H be a graph on which Maker can win the triangle game. If H has an edge that does not participate in a triangle, then we can remove that edge and Maker can still win. Next, if H has a vertex v of degree 2 or less, then we can remove that vertex and Maker will still be able to win. To see this, assume for a contradiction that Breaker has a winning strategy on $H - v$. If $d(v) \leq 1$, then v does not participate in a triangle, so Breaker can trivially win on H as well, a contradiction. If $d(v) = 2$, then Breaker's winning strategy is the following – he pairs the two edges at v (so that when Maker claims one of them, he immediately claims the other one), and he plays on $H - v$ according to the mentioned winning strategy, which again gives a contradiction.

Looking at H, we build an auxiliary graph G_H whose vertices are all the triangles of H, where two triangles form an edge in G_H if and only if they share an edge in H. Now, we claim that there is a connected component C of G_H so that Maker can win the game on the union of all triangles in C. Indeed, the triangles in different components of G_H have no edges in common. So, if Breaker had a winning strategy on every connected component of G_H, then he could combine those strategies to win the game on H. But, we know that Maker wins the game on H. If G_H is connected, we will call H *triangle-connected.*

Now, as Maker can win the triangle game on G_{t_1}, we know that G_{t_1} has to contain a subgraph G' which satisfies the following properties:

- Every edge of G' participates in a triangle, and;
- $\delta(G') \geq 3$, and;
- G' is triangle-connected, and;
- Maker wins the triangle game on G'.

To bound the number of vertices of the graph G', we will use the approach similar to the one used in [20]. We initially define $T_{G'}$ to be the set containing an arbitrary triangle T_1 of G', and then we enlarge it by repeatedly adding triangles of G' that have exactly one vertex that is not contained in any triangle in $T_{G'}$ for as long as possible. Once this process is complete, the union of vertex sets of all triangles in $T_{G'}$ must be $V(G')$. Indeed, assume for a contradiction that some $v \in V(G')$ is contained in no triangle in $T_{G'}$. Since

$\delta(G') > 2$ and every edge of G' participates in a triangle, there must be a triangle T_2 in G' containing v. From what we assumed, T_2 has a vertex not contained in any triangle in $T_{G'}$, and in particular $T_2 \notin T_{G'}$. Then, knowing that G' is triangle-connected, there is a sequence of triangles of G', starting with T_1 and ending with T_2, such that every two consecutive triangles share an edge. Since $T_1 \in T_{G'}$, we can find the first triangle in the sequence that has a vertex not contained in any triangle in $T_{G'}$, we denote it by T_0. But as T_0 shares an edge with a triangle that does not satisfy this property, it must have *exactly* one vertex not contained in any triangle in $T_{G'}$, which means that it could be added to $T_{G'}$ according to the adding rule, a contradiction.

If we denote $\ell := v(T_{G'})$, then the graph $G'' := \bigcup_{T \in T_{G'}} T$ has $\ell + 2$ vertices and at least $2\ell + 1$ edges. As we already saw, G'' is a spanning subgraph of G'. If $\ell \geq 14$, then the maximum density of G' is at least $29/16 > 9/5$. We know that there will be no graph on 14 vertices with such density in G_{t_1} a.a.s., since $t_1 \leq t_2 \leq n^{13/9} \log n$, see [14]. Hence, G' has at most 13 vertices a.a.s.

Let $d = e(G') - (2\ell + 1)$. We distinguish four cases.

(i) $d = 0$

Since $e(G'') \geq 2\ell + 1$, we have $G' = G''$. But this is not possible, as the triangle that is added last to $T_{G'}$ ensures that there is a vertex of G'' of degree 2, and the minimum degree of G' is at least 3.

(ii) $d = 1$

Now we have $2\ell + 2 \geq e(G') \geq e(G'') \geq 2\ell + 1$. By the argument just given for the case (i), we cannot have that $G' = G''$. We conclude that $e(G'') = 2\ell + 1$. Let $\bar{e}$ denote the edge of $E(G') \setminus E(G'')$. Since $e(G'') = 2\ell + 1$, every new triangle added to $T_{G'}$, except the first one, contributed exactly two new edges to G''. A straightforward inductive argument now shows that G'' has at least two non-adjacent vertices of degree two (except if $\ell = 1$ – in which case both G' and G'' are triangles which is clearly impossible).

Since G' has just one edge more than G'', and $\delta(G') \geq 3$, clearly G'' must have *exactly* two vertices of degree 2. Moreover, those two vertices must have a common neighbor, as otherwise the edge $\bar{e}$ would not participate in any triangle. The only way this can happen is when G' is a wheel. But, it is easy to see that Breaker can win the triangle game on any wheel using a simple pairing strategy. Hence, this case also leads to a contradiction.

(iii) $d = 2$

The maximum density of G' in this case is at least $\frac{2\ell+3}{\ell+2}$. For $\ell > 3$ this value is over $9/5$, so, arguing similarly as before, this subcase will not happen a.a.s. Hence, $\ell \leq 3$, and G' has at most 5 vertices. The only way Maker can win the triangle game on G' with these restrictions is if G' is $K_5 - e$, and that gives $t_1 = t_2$.

(iv) $d > 2$

The maximum density of G' in this case is at least $\frac{2\ell+4}{\ell+2} = 2 > 9/5$. Similarly as before, this case will not happen a.a.s.

Hence, we have that $t_1 = t_2$ a.a.s. ■

4 Conclusion and open problems

Random graph intuition. Chvátal and Erdős [9] observed the following paradigm, which is referred to as the *random graph intuition* in positional game theory. As it turns out for many standard games on graphs, the inverse of the threshold bias $b_{\mathcal{F}}$ in the game played on the complete graph is "closely related" to the probability threshold for the appearance of a member of $\mathcal{F}$ in $G(n,p)$. Another parameter that is often "around" is the threshold probability $p_{\mathcal{F}}$ for Maker's win when played on $G(n,p)$. As we saw, for the two games mentioned in the introduction, the connectivity game and the Hamilton cycle game, all three parameters are exactly equal to $\frac{\log n}{n}$.

In the k-clique game, for $k \geq 4$, the threshold bias is $b_{\mathcal{K}_k} = \Theta(n^{\frac{2}{k+1}})$ and the threshold probability for Maker's win is the inverse (up to the leading constant), $p_{\mathcal{K}_k} = n^{-\frac{2}{k+1}}$, supporting the random graph intuition. But, the threshold probability for appearance of a k-clique in $G(n,p)$ is not at the same place, it is $n^{-\frac{2}{k-1}}$. And in the triangle game there is even more disagreement, as all three parameters are different – they are, respectively, $n^{\frac{1}{2}}$, $n^{-\frac{5}{9}}$ and n^{-1}. Now, more than thirty years after Chvátal and Erdős formulated the paradigm, there is still no general result that would make it more formal. We are curious to the reasons behind the total agreement between the three thresholds in the connectivity game and the Hamilton cycle game, partial disagreement in k-clique game for $k \geq 4$, and the total disagreement in the triangle game.

Random clique game vs. biased clique game. Our Corollary 1.2 gives two constants $c > 0$ and $C > 0$, stating that the probability threshold for Maker's win in the k-clique game on $G(n,p)$ for $k \geq 4$ is between $cn^{-\frac{2}{k+1}}$ and $Cn^{-\frac{2}{k+1}}$. In a way, with this result, the game played on the random graph catches up with the biased k-clique game played on the complete graph, as a result of Bednarska and Łuczak [4] guarantees the existence of constants $c' > 0$ and $C' > 0$, such that the bias threshold for this game is between $c'n^{\frac{2}{k+1}}$ and $C'n^{\frac{2}{k+1}}$, for all $k \geq 3$. Both pairs of constants, c, C and c', C', are quite far apart. Also, in both games, the best known strategy for Maker's exploits the same derandomized random strategy approach, proposed in [4].

We know much more for the triangle game on the random graph, as Corollary 1.4 gives the threshold probability quite accurately, and it turns out to be a coarse threshold. The reason for such different behavior (compared to $k > 3$) may lie behind the fact that $K_3 = C_3$. We note that the triangle game is a representative of almost disjoint 3-regular positional games which were analyzed in [17], but we do not know enough about the winning set structure in our case to apply the results from [17].

A more precise result for the k-clique game when $k \geq 4$? As we saw, we can say a lot about the threshold probability for the triangle game, the connectivity game and the Hamilton cycle game when the game is played on the random graph. We do not know that much about the k-clique game, when $k \geq 4$, and it would be very interesting to see what happens between the bounds given in Corollary 1.2. Also, a graph-theoretic description of the hitting time of Maker's win on the random graph process would be of great importance, as currently we know very little about Maker's winning strategy at the threshold. What we know is that we cannot hope for a result analogous to Theorem 1.3 – the reason for Maker's win cannot be the appearance of a fixed graph, as Lemma 2.1 guarantees Breaker's win on every typical (fixed) subgraph of the random graph on the probability threshold. Hence, Maker's optimal strategy must be of "global nature", taking into account a non-constant part of the random graph to win the game. Having that in mind we propose the following conjecture.

Conjecture 4.1 *For every $k \geq 4$ there exists a $c = c(k)$ such that for any fixed $\varepsilon > 0$:*

(i) *If $p \leq (c-\varepsilon)n^{-\frac{2}{k+1}}$, then Breaker wins the k-clique game on $G(n,p)$ a.a.s;*

(ii) *If $p \geq (c+\varepsilon)n^{-\frac{2}{k+1}}$, then Maker wins the k-clique game a.a.s.*

It might be possible to apply the celebrated results of Friedgut [10] to get something slightly weaker than the conjecture. We have however not been able to make such an argument stick.

H **game.** A natural extension of the clique game is the H game, where Maker's goal is to claim a copy of a given graph H. For the game played on the random graph, we know much less in this case. Some progress has been made in [19]. Apart from results about the threshold probability and hitting time results, a characterization of all graphs H for which the threshold probability in the game on the random graph is equal to the inverse of the threshold bias in the game on the complete graph would be of considerable importance.

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