

On the contractibility of random Vietoris–Rips complexes

Tobias Müller*

Matěj Stehlík†

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Abstract

We show that the Vietoris–Rips complex $\mathcal{R}(n, r)$ built over n points sampled at random from a uniformly positive probability measure on a convex body $K \subseteq \mathbb{R}^d$ is a.a.s. contractible when $r \geq c \left(\frac{\ln n}{n}\right)^{1/d}$ for a certain constant that depends on K and the probability measure used. This answers a question of Kahle [14]. We also extend the proof to show that if K is a compact, smooth d -manifold with boundary – but not necessarily convex – then $\mathcal{R}(n, r)$ is a.a.s. homotopy equivalent to K when $c_1 \left(\frac{\ln n}{n}\right)^{1/d} \leq r \leq c_2$ for constants $c_1 = c_1(K), c_2 = c_2(K)$. Our proofs expose a connection with the game of cops and robbers.

1 Introduction

At least in part because of their role as null models in topological data analysis, simplicial complexes built on random points in d -dimensional Euclidean space have attracted a lot of attention over the past decade or so. We refer the reader to the recent survey article [8] and the references therein for more background information and an overview of the results on random geometric complexes.

In this note, $K \subseteq \mathbb{R}^d$ will either be a convex body (a convex, compact set with nonempty interior) or a compact, smooth d -manifold with boundary; and ν will be a probability measure on \mathbb{R}^d with probability density f , whose support is K and which is uniformly positive on K . (I.e. $\inf_{x \in K} f(x) > 0$.) We will consider the random Vietoris–Rips complex $\mathcal{R}(n, r)$ constructed by sampling X_1, \dots, X_n i.i.d. according to ν and declaring a subset of these points a simplex if and only if all its pairwise distances are at most r .

If $(E_n)_n$ is a sequence of events then we say that E_n holds *asymptotically almost surely* (a.a.s.) if $\mathbb{P}(E_n) \rightarrow 1$. Our first result is as follows.

Theorem 1 *If $K \subseteq \mathbb{R}^d$ is a convex body then there is a constant $c = c(K, \nu)$ such that $\mathcal{R}(n, r_n)$ is a.a.s. contractible whenever $r_n \geq c \left(\frac{\ln n}{n}\right)^{1/d}$.*

This theorem answers a question of Kahle [14] (bottom of page 569), who asked whether Theorem 1 holds in the case when K has smooth boundary in addition to being a convex body and ν is the uniform distribution on K .

*Bernoulli Institute, Groningen University, The Netherlands. E-mail: tobias.muller@rug.nl. Partially supported by NWO grants 639.032.529 and 612.001.409.

†Universit  de Paris, CNRS, IRIF, F-75006, Paris, France. E-mail: matej@irif.fr. This work was commenced while the author was at Laboratoire G-SCOP, Univ. Grenoble Alpes, France. Partially supported by ANR project GATO (ANR-16-CE40-0009-01) and by LabEx PERSYVAL-Lab (ANR-11-LABX-0025).

If we assume K is smoothly bounded then we can let go of the condition that K be convex and obtain the following. Here and in the rest of the paper we use \simeq to denote homotopy equivalence.

Theorem 2 *If $K \subseteq \mathbb{R}^d$ is a compact, smooth d -manifold with boundary then there are constants $c_1 = c_1(K, \nu), c_2 = c_2(K)$ such that $\mathcal{R}(n, r_n) \simeq K$ a.a.s. whenever $c_1 \left(\frac{\ln n}{n}\right)^{1/d} \leq r_n \leq c_2$.*

Note that the condition $r_n \leq c_2(K)$ is in general necessary. If for instance $r \geq \text{diam}(K)$ then $\mathcal{R}(n, r_n)$ will be the complete simplicial complex on n vertices, and in particular contractible, regardless of the precise homotopy type of K . For clarity, we emphasize that in both Theorem 1 and 2, the metric used for the construction of the Vietoris–Rips complex is the Euclidean metric on the ambient space \mathbb{R}^d .

Related work. A widely studied subcomplex of the Vietoris–Rips complex is the Čech complex, where a set of points spans a simplex if and only if the balls of radius $r/2$ around them have a non-empty intersection. Niyogi, Smale and Weinberger [21] obtained a result for random Čech complexes that is similar to Theorem 2, and Kahle [14] proved a statement analogous to Theorem 1 for random Čech complexes under the uniform probability measure. Homological connectivity is a notion closely related to contractibility. Results on homological connectivity of random Čech complexes can be found in [7, 11, 12, 14].

There is a substantial literature on abstract combinatorial models of random simplicial complexes, including the Linal-Meshulam model introduced by Linial and Meshulam in [17], the random d -complex introduced by Meshulam and Wallach in [19], and the random clique complex introduced by Kahle in [13]. The random clique complex is the clique complex (see the next section for the precise definition) of the Erdős–Rényi random graph and thus in a sense analogous to the random Vietoris–Rips complex, which is the clique complex of the random geometric graph. The papers of Kahle [13, 15] and Malen [18] contain results in the same spirit as ours for this model.

2 Notation and preliminaries

Here we list some notations, definitions and results we will use in the proofs. For $k \in \mathbb{N}$ a positive integer we write $[k] := \{1, \dots, k\}$.

A *simplicial complex* is a pair $\Delta = (V, \Sigma)$ with V a finite set and $\Sigma \subseteq 2^V$ closed under taking subsets. That is, the elements of Σ are subsets of V and $\sigma \in \Sigma$ and $\tau \subseteq \sigma$ implies that also $\tau \in \Sigma$. The elements of V are called vertices and the elements of Σ simplices. The standard geometric realization of a complex $\Delta = (V, \Sigma)$ is given by

$$\|\Delta\| := \bigcup_{\sigma \in \Sigma} \text{conv}(\{e_i : v_i \in \sigma\}),$$

where e_1, \dots, e_n denote the standard basis vectors for \mathbb{R}^n with $n := |V|$ and we fix some enumeration $V = \{v_1, \dots, v_n\}$. A topological space X is *triangulable* if there exists a simplicial complex Δ such that X is homeomorphic to $\|\Delta\|$. We remark that if $K \subseteq \mathbb{R}^d$ is a convex body then K is homeomorphic to the convex hull of the standard basis in \mathbb{R}^{d+1} and in particular triangulable. If K is a compact, smooth d -manifold with boundary then K is also triangulable by standard results in topology (see for instance [20], Theorem 10.6).

If X, Y are two topological spaces and $f, g : X \rightarrow Y$ continuous maps then f and g are *homotopic* (denoted $f \simeq g$) if there exists a continuous map $\varphi : X \times [0, 1] \rightarrow Y$ such that $\varphi(\cdot, 0) = f$ and $\varphi(\cdot, 1) = g$. The spaces X, Y are *homotopy equivalent* (notation $X \simeq Y$) if there exists maps $f_1 : X \rightarrow Y, f_2 : Y \rightarrow X$ such that $f_2 \circ f_1 \simeq \text{id}_{X_1}$ and $f_1 \circ f_2 \simeq \text{id}_{X_2}$. We say X is *contractible* if it is homotopy equivalent to a single point. If X is a topological space and Δ a simplicial complex then $X \simeq \Delta$ denotes that X and $\|\Delta\|$ are homotopy equivalent. Similarly, for Δ_1, Δ_2 simplicial complexes $\Delta_1 \simeq \Delta_2$ denotes their geometric realizations are homotopy equivalent. We say Δ is contractible if its geometric realization $\|\Delta\|$ is.

The *nerve* of a family of sets $\mathcal{A} = (A_i)_{i \in I}$ is the simplicial complex $\mathcal{N} = \mathcal{N}(\mathcal{A})$ with vertex set I where a finite set $\sigma \subseteq I$ is a simplex of \mathcal{N} if and only if $\bigcap_{i \in \sigma} A_i \neq \emptyset$. We will use two different versions of the *nerve theorem*, stated as Theorems 10.6 and 10.7 in [6].

Theorem 3 (Nerve theorem, combinatorial version) *Let Δ be a simplicial complex and let $(\Delta_i)_{i \in I}$ be a family of subcomplexes such that $\Delta = \bigcup_{i \in I} \Delta_i$, and every nonempty finite intersection $\Delta_{i_1} \cap \dots \cap \Delta_{i_k}$ is contractible. Then $\mathcal{N}(\Delta) \simeq \Delta$.*

Theorem 4 (Nerve theorem, geometric version) *Let X be a triangulable space and $\mathcal{A} = (A_i)_{i \in I}$ either a locally finite family of open subsets or a finite family of closed subsets, such that $X = \bigcup_{i \in I} A_i$. If every nonempty finite intersection $A_{i_1} \cap \dots \cap A_{i_k}$ is contractible, then $\mathcal{N}(\mathcal{A}) \simeq X$.*

For G a graph and $v \in V(G)$ a vertex of G we denote by $G \setminus v$ the graph with the vertex v and all incident edges removed. By $N_G(v)$ we denote the set of neighbours of v and by $N_G[v] := N_G(v) \cup \{v\}$ its *closed neighbourhood*. If no confusion can arise we drop the subscript and simply write $N(v), N[v]$.

If G is a graph, then $\mathcal{K}(G)$ denotes its *clique complex*, or *flag complex*. That is, $\mathcal{K}(G)$ has vertex set $V(G)$ and a subset of the vertices is declared a simplex if and only if it spans a complete subgraph of G . So in particular, $\mathcal{R}(n, r) = \mathcal{K}(G(n, r))$ where $G(n, r)$ denotes the random geometric graph with vertices X_1, \dots, X_n and an edge $X_i X_j$ if and only if $\|X_i - X_j\| \leq r$.

We will use $B(x, r) := \{y \in \mathbb{R}^d : \|x - y\| < r\}$ to denote the open ball of radius r around x . We use $\text{vol}(\cdot)$ to denote the d -dimensional volume (Lebesgue measure) and we denote by $\pi_d := \text{vol}(B(\mathbf{0}, 1))$ the volume of the d -dimensional unit ball. For $X \subseteq \mathbb{R}^d$ we denote its convex hull by $\text{conv}(X)$, its *diameter* by $\text{diam}(X) := \sup\{\|x - y\| : x, y \in X\}$ and we define its *inradius* by

$$\text{inr}(X) := \sup\{r : \exists x \text{ such that } B(x, r) \subseteq X\}.$$

3 Proofs

3.1 The proof of Theorem 1

We start with the following key observation.

Lemma 5 *If G is a graph and $v \neq w \in V(G)$ are such that $N[v] \subseteq N[w]$ then $\mathcal{K}(G) \simeq \mathcal{K}(G \setminus v)$.*

Proof. Without loss of generality we can assume $V(G) = [n], v = n, w = n - 1$. We consider the standard geometric realizations $X_1 := \|\mathcal{K}(G)\|, X_2 := \|\mathcal{K}(G \setminus v)\|$ where we identify \mathbb{R}^{n-1} with $\mathbb{R}^{n-1} \times \{0\} \subseteq \mathbb{R}^n$ so that $X_2 \subseteq X_1$. Put differently, we have

$$X_1 = \bigcup_{\substack{A \subseteq [n], \\ A \text{ a clique in } G}} \text{conv}(\{e_i : i \in A\}), \quad X_2 = \bigcup_{\substack{A \subseteq [n] \setminus \{v\}, \\ A \text{ a clique in } G}} \text{conv}(\{e_i : i \in A\}),$$

with e_1, \dots, e_n the standard basis for \mathbb{R}^n . We need to demonstrate the existence of maps $f_1 : X_1 \rightarrow X_2$ and $f_2 : X_2 \rightarrow X_1$ such that $f_1 \circ f_2$ is homotopic to the identity on X_2 and $f_2 \circ f_1$ is homotopic to the identity on X_1 . We define these maps by

$$f_1(x_1, \dots, x_n) := (x_1, \dots, x_{n-2}, x_{n-1} + x_n, 0), \quad f_2(x) := x.$$

So $f_2 : X_2 \rightarrow X_1$ is simply “the inclusion map”, but we need to verify that $f_1(x) \in X_2$ for every $x \in X_1$. To this end, we first note that if $x = (x_1, \dots, x_n)$ satisfies $x_n = 0$ then $f_1(x) = x \in X_2 \subseteq X_1$. Suppose then $x_n \neq 0$ and let $A := \text{supp}(x) = \{i : x_i \neq 0\}$ denote the support of x . By definition of X_1 , we have that A is a clique of G and $\sum_{i \in A} x_i = 1$ and $x_i > 0$ for all $i \in A$. Setting $y := f_1(x)$ we have that $B := \text{supp}(y) = (A \cup \{n-1\}) \setminus \{n\}$. By definition of f_1 and A and B

$$\sum_{i \in B} y_i = \sum_{i \in A} x_i = 1,$$

and $y_i > 0$ for all $i \in B$. So we just need to establish that B is a clique of $G \setminus n$ in order to verify that $y \in X_2$. In fact, it suffices to show $A \cup \{n-1\}$ is a clique of G . To see this, we remark that A is a clique in G with

$$A \subseteq N[n] \subseteq N[n-1],$$

using $x_n \neq 0$ for the first inclusion. In particular each element of A is either a neighbour of $n-1$ or $n-1$ itself. It follows that $A \cup \{n-1\}$ is a clique indeed.

Since $x_n = 0$ whenever $(x_1, \dots, x_n) \in X_2$, we have $f_1 \circ f_2 = \text{id}_{X_2}$. It remains to see that $f_2 \circ f_1 \simeq \text{id}_{X_1}$. We define $\varphi : X_1 \times [0, 1] \rightarrow X_1$ via

$$\varphi(x_1, \dots, x_n, t) := (x_1, \dots, x_{n-2}, x_{n-1} + tx_n, (1-t)x_n).$$

The map φ satisfies $\varphi(\cdot, 0) = \text{id}_{X_1}, \varphi(\cdot, 1) = f_2 \circ f_1$ and is obviously continuous, but we need to establish that $\varphi(x, t) \in X_1$ for all $t \in [0, 1]$ and $x \in X_1$. To this end, let $x = (x_1, \dots, x_n) \in X_1$ and $t \in [0, 1]$ be arbitrary. If $x_n = 0$ then $\varphi(x, t) = x$. Let us thus assume $x_n \neq 0$, and again set $A := \text{supp}(x), y := \varphi(x, t), B := \text{supp}(y)$. By definition of φ we again have $B \subseteq A \cup \{n-1\}$ and $\sum_{i \in B} y_i = 1$ and $y_i > 0$ for all $i \in B$. Repeating previous arguments, we find that B is a clique of G and hence $y \in X_1$. \blacksquare

For $x, y \in \mathbb{R}^d, r > 0$ we define:

$$W(x, y, r) := \{z \in B(y, \|y - x\|) : B(z, r) \supseteq B(x, r) \cap B(y, \|y - x\|)\}. \quad (1)$$

See Figure 1 for a depiction. The idea behind this definition is as follows. Suppose x, y are vertices of a subgraph H of the random geometric graph, and suppose that x is the vertex of H that is furthest from y . Then any vertex in $z \in W(x, y, r) \cap V(H)$ satisfies $N_H[z] \supseteq N_H[x]$.

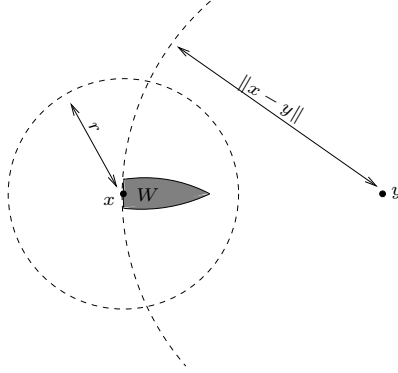


Figure 1: Depiction of the set $W(x, y, r)$, in the two-dimensional case.

Lemma 6 *For every $\lambda > 0$ there exists a $\delta_1 = \delta_1(\lambda) > 0$ such that for all $r > 0$ and $x, y \in \mathbb{R}^d$ with $r \leq \|x - y\| \leq \lambda r$, there is a $z \in \text{conv}(\{x, y\})$ such that $B(z, \delta_1 r) \subseteq W(x, y, r)$.*

Proof. By applying a suitable dilation and rigid motion if needed, we can assume without loss of generality that $r = 1, x = \mathbf{0}, y = (y_1, 0, \dots, 0)$ with $1 \leq y_1 \leq \lambda$.

We set

$$z := \left(\frac{1}{10\lambda}, 0, \dots, 0 \right), \quad \delta_1 := \min \left(\frac{1}{10\lambda}, 1 - \left(1 - \frac{1}{100\lambda^2} \right)^{1/2} \right).$$

We observe that, since $0 < 1/(10\lambda) < \lambda$, we have $z \in \text{conv}(\{x, y\})$; and by the choice of δ_1 we have $B(z, \delta_1) \subseteq B(y, \|y - x\|)$.

Pick an arbitrary $u = (u_1, \dots, u_d) \in B(x, 1) \cap B(y, \|y - x\|)$. To complete the proof, we need to show $u \in B(z', 1)$ for every $z' \in B(z, \delta_1)$. We remark that it in fact suffices to show that $\|u - z\| < \left(1 - \frac{1}{100\lambda^2} \right)^{1/2}$. In order to prove that, we consider two cases. We first suppose that $u_1 \geq z_1 = \frac{1}{10\lambda}$. Since $\|u - \mathbf{0}\| < 1$, we have $\sum_{i \geq 2} u_i^2 < 1 - u_1^2$. Hence

$$\|u - z\|^2 = (u_1 - z_1)^2 + \sum_{i \geq 2} u_i^2 < (u_1 - z_1)^2 + 1 - u_1^2 = 1 + z_1^2 - 2u_1 z_1 \leq 1 - z_1^2 = 1 - \frac{1}{100\lambda^2}.$$

Let us now suppose that $u_1 < z_1$. Since $u \in B(y, \|y - x\|) = B(y, \|y\|)$ we have $u_1 > 0$ and $\|u - y\| < \|y\|$. Therefore

$$\sum_{i \geq 2} u_i^2 < y_1^2 - (y_1 - u_1)^2 < y_1^2 - (y_1 - z_1)^2 = 2y_1 z_1 - z_1^2 \leq 2\lambda z_1 - z_1^2,$$

giving

$$\|u - z\|^2 = (u_1 - z_1)^2 + \sum_{i \geq 2} u_i^2 < z_1^2 + 2\lambda z_1 - z_1^2 = 2\lambda z_1 = \frac{1}{5} < 1 - \frac{1}{100\lambda^2},$$

using $0 < u_1 < z_1$ in the first inequality and $\lambda \geq 1$ in the second inequality. \blacksquare

The only probabilistic ingredient we will need is the lemma below on the probability that the balls of radius r around the random points X_1, \dots, X_n cover K .

Lemma 7 *Suppose $K \subseteq \mathbb{R}^d$ is a convex body. There is a $\delta_2 = \delta_2(K, \nu)$ such that if $r_n \geq \delta_2 \cdot \left(\frac{\ln n}{n}\right)^{1/d}$ then, a.a.s., $K \subseteq \bigcup_{i=1}^n B(X_i, r_n)$.*

We will employ the following observation in the proof of Lemma 7 that will be of use to us later on as well.

Lemma 8 *For every $\lambda > 0$ there exists a $\delta_3 = \delta_3(\lambda) > 0$ such that, for every convex $X \subseteq \mathbb{R}^d$ with $\text{diam}(X) \leq \lambda \cdot \text{inr}(X)$, for every $0 < r \leq \text{diam}(X)$ and $x \in X$ there is a z such that*

$$B(z, \delta_3 r) \subseteq B(x, r) \cap X.$$

Proof. Applying a suitable dilation if needed, we can assume without loss of generality $\text{inr}(X) = 1$. By assumption, there exists $y \in X$ such that $B(y, 1/2) \subseteq X$. Applying a suitable translation if needed, we can assume without loss of generality that $y = \underline{0}$ is the origin. By convexity, the set

$$B_\mu := \{\mu z + (1 - \mu)x : z \in B(\underline{0}, 1/2)\}.$$

is contained in X for every $0 \leq \mu \leq 1$. On the other hand B_μ is a ball of radius $\mu/2$ centred at the point $(1 - \mu)x$. In particular

$$B_\mu \subseteq B(x, \mu/2 + \|x - (1 - \mu)x\|) = B(x, \mu \cdot (1/2 + \|x\|)) \subseteq B(x, \mu \cdot (1/2 + \lambda)).$$

So, if we set $\mu := r/(1/2 + \lambda)$ then $B_\mu \subseteq B(x, r) \cap X$ is a ball of radius $\delta_3 r$ where $\delta_3 = \delta_3(\lambda) := 1/(1 + 2\lambda)$. ■

Proof of Lemma 7. We set $r := \delta_2 \cdot (\ln n/n)^{1/d}$ where the constant δ_2 will be determined in the course of the proof. We write $s := r/4$ for convenience. Next, we pick $x_1, \dots, x_N \in K$ such that $B(x_1, s), \dots, B(x_N, s)$ are disjoint and $K \subseteq \bigcup_{i=1}^N B(x_i, 2s)$. (Such x_1, \dots, x_N can for instance be found by “greedily” constructing a maximal packing of balls of radius s with centers in K .) By Lemma 8, for each i , $B(x_i, s) \cap K$ contains a ball of radius $\delta_3 \cdot s$ with $\delta_3 = \delta_3(\text{diam}(K)/\text{inr}(K))$. Hence, writing $\nu_{\min} := \inf_{x \in K} f(x)$, we have, for each $i = 1, \dots, N$:

$$\nu(B(x_i, s)) \geq \nu_{\min} \cdot \pi_d \cdot (\delta_3 s)^d = \nu_{\min} \cdot \pi_d \cdot (\delta_2/4)^d \cdot \delta_3^d \cdot \left(\frac{\ln n}{n}\right)^d. \quad (2)$$

Since the balls $B(x_i, s)$ are disjoint, we have $1 \geq \sum_i \nu(B(x_i, s))$. Combining this with (2) gives $N = O(n/\ln n)$. The inequality (2) also implies that

$$\begin{aligned} \mathbb{P}(B(x_i, 2s) \cap \{X_1, \dots, X_n\} = \emptyset) &= (1 - \nu(B(x_i, 2s)))^n \\ &\leq \left(1 - \nu_{\min} \cdot \pi_d \cdot \delta_2^d \cdot \delta_3^d \cdot \left(\frac{\ln n}{n}\right)^d\right)^n \\ &\leq \exp\left[-\ln n \cdot \nu_{\min} \cdot \pi_d \cdot (\delta_2/4)^d \cdot \delta_3^d\right], \end{aligned}$$

for each i . As $K \subseteq \bigcup_{i=1}^N B(x_i, 2s)$ and $r = 4s$ we see that

$$\begin{aligned} \mathbb{P}(K \text{ is not covered by } \bigcup_{i=1}^n B(x_i, r)) &\leq \mathbb{P}(B(x_i, 2s) \cap \{X_1, \dots, X_n\} = \emptyset \text{ for some } 1 \leq i \leq N) \\ &\leq N \cdot \exp\left[-\ln n \cdot \nu_{\min} \cdot \pi_d \cdot (\delta_2/4)^d \cdot \delta_3^d\right] \\ &= o(1), \end{aligned}$$

where the last line holds provided we chose $\delta_2 > 4^d / (\nu_{\min} \cdot \pi_d \cdot \delta_3^d)$. \blacksquare

Lemma 9 *Let $K \subseteq \mathbb{R}^d$ be a convex body. There exists a constant $\delta_4 = \delta_4(K)$ such that, for every $0 < r \leq \text{diam}(K)$, there exists a finite family of sets $A_1, \dots, A_N \subseteq K$ such that*

- (i) *Each A_i is open in the relative topology of K , and;*
- (ii) *For every $S \subseteq K$ with $\text{diam}(S) \leq r$ there is a $1 \leq i \leq N$ such that $S \subseteq A_i$, and;*
- (iii) *For every $I \subseteq [N]$, if $\bigcap_{i \in I} A_i$ is nonempty then it is contractible, and;*
- (iv) *For every $I \subseteq [N]$, if $\bigcap_{i \in I} A_i$ is nonempty then it contains a ball of radius $\delta_4 \cdot r$, and;*
- (v) *For every $I \subseteq [N]$, if $x, y \in \bigcap_{i \in I} A_i$ satisfy $\|x - y\| \geq r$ then $W(x, y, r) \cap (\bigcap_{i \in I} A_i)$ contains a ball of radius $\delta_4 \cdot r$.*

Proof. We let $x_1, \dots, x_N \in K$ be such that $B(x_1, r), \dots, B(x_N, r)$ are disjoint and $K \subseteq \bigcup_{i=1}^N B(x_i, 2r)$. We are going to set $A_i := B(x_i, s_i) \cap K$ where the radii $3r \leq s_i \leq 4r$ will be determined via an iterative procedure. Initially we set $s_1 = \dots = s_N = 3r$. We let $\varepsilon > 0$ be a parameter, to be chosen more precisely later on.

If there is a set of indices $I \subseteq [N]$ such that $K \cap \bigcap_{i \in I} B(x_i, s_i)$ is nonempty, but there is no $x \in K \cap \bigcap_{i \in I} B(x_i, s_i)$ such that $B(x, \varepsilon r) \subseteq \bigcap_{i \in I} B(x_i, s_i)$, then we increase s_i by εr for each $i \in I$. (Note that this will ensure that there is an $x \in K$ with $B(x, \varepsilon r) \subseteq \bigcap_{i \in I} B(x_i, s_i)$.) We keep repeating this step until this is no longer possible.

Since the operation is applied at most once to each $I \subseteq [N]$ the procedure finishes after a finite number of operations. It may however not be obvious that there is a choice of $\varepsilon > 0$ for which this procedure will finish in a situation where $s_1, \dots, s_N \leq 4r$. To see this we set

$$I_i := \{j : B(x_i, 4r) \cap B(x_j, 4r) \neq \emptyset\}, \quad m_i := |I_i|.$$

If $B(x_i, 4r) \cap B(x_j, 4r) \neq \emptyset$ then $B(x_j, r) \subseteq B(x_i, 9r)$. As the balls $B(x_1, r), \dots, B(x_N, r)$ are disjoint this gives

$$\pi_d \cdot (9r)^d = \text{vol}(B(x_i, 9r)) \geq \sum_{j \in I_i} \text{vol}(B(x_j, r)) = m_i \cdot \pi_d \cdot r^d.$$

It follows that

$$m_i \leq 9^d.$$

The number of sets of indices I with $i \in I$ and $\bigcap_{j \in I} B(x_j, 4r) \neq \emptyset$ is therefore at most $2^{m_i} \leq 2^{9^d}$. Having chosen $\varepsilon := \frac{1}{100} \cdot 2^{-9^d}$ we see that no radius s_i will ever exceed $4r$.

We need to prove that, for a suitable choice of the constant δ_4 , the constructed sets $A_1 := B(x_1, s_1) \cap K, \dots, A_N := B(x_N, s_N) \cap K$ satisfy the properties claimed by the lemma. That (i) holds is obvious and that (iii) holds follows immediately from the fact that each A_i , and hence also each nonempty intersection $\bigcap_{i \in I} A_i$, is convex. To see that (ii) holds, let $S \subseteq K$ with $\text{diam}(S) \leq r$ be arbitrary and fix $s \in S$. Since $K \subseteq B(x_1, 2r) \cup \dots \cup B(x_N, 2r)$ there is some i such that $s \in B(x_i, 2r)$. But then

$$S \subseteq K \cap B(s, r) \subseteq K \cap B(x_i, 3r) \subseteq K \cap B(x_i, s_i) = A_i.$$

It remains to establish **(iv)**, **(v)** for a suitable choice of the constant δ_4 . We let $\delta_3 = \delta_3(\text{diam}(K)/\text{inr}(K))$ be as provided by Lemma 8. If I is such that $\bigcap_{i \in I} A_i = K \cap \bigcap_{i \in I} B(x_i, s_i)$ is non-empty then by construction there exists $x \in K$ such that $B(x, \varepsilon r) \subseteq \bigcap_{i \in I} B(x_i, s_i)$. Applying Lemma 8 we find a ball $B(z, \delta_3 \varepsilon r) \subseteq B(x, \varepsilon r) \cap K \subseteq \bigcap_{i \in I} A_i$.

Now suppose $x, y \in \bigcap_{i \in I} A_i$ with $\|x - y\| \geq r$, and let $\delta_1 = \delta_1(8/\delta_3 \varepsilon)$ be as provided by Lemma 6. (Observe that $\text{diam}(\bigcap_{i \in I} A_i) \leq 8r$ and $\text{inr}(\bigcap_{i \in I} A_i) \geq \delta_3 \varepsilon r$.) It follows from Lemma 6 that there exists a ball $B(z, \delta_1 r) \subseteq W(x, y, r)$ with $z \in \text{conv}(\{x, y\}) \subseteq \bigcap_{i \in I} A_i$. (The inclusion holding by convexity.) Applying Lemma 8 again, we find a ball

$$B(z', \delta_3 \delta_1 r) \subseteq B(z, \delta_1 r) \cap \bigcap_{i \in I} A_i \subseteq W(x, y, r) \cap \bigcap_{i \in I} A_i.$$

We conclude that **(iv)** and **(v)** hold with $\delta_4 := \min(\delta_3 \delta_1, \delta_3 \varepsilon)$. ■

Proof of Theorem 1. By Lemma 7, there is a $\delta_2 = \delta_2(K)$ such that, a.a.s., every ball $B(x, \delta_2 (\ln n/n)^{1/d})$ with $x \in K$ contains at least one of the random points X_1, \dots, X_n . We set $c := \delta_2/\delta_4$ with $\delta_4 = \delta_4(K)$ as provided by Lemma 9. Let $r \geq c (\ln n/n)^{1/d}$ be arbitrary. Note that if $r \geq \text{diam}(K)$ then $\mathcal{R}(n, r)$ is the complete simplicial complex on n vertices and we are trivially done. We thus assume $r < \text{diam}(K)$ for the remainder of the proof, and we let A_1, \dots, A_N be as provided by Lemma 9. By the geometric version of the nerve theorem and items **(i)** and **(iii)** of Lemma 9, we have $\mathcal{N}((A_i)_{i=1, \dots, N}) \simeq K$.

Let Δ_i denote subcomplex of $\mathcal{R}(n, r)$ induced by the points $\{X_1, \dots, X_n\} \cap A_i$ that fall in A_i . By the combinatorial version of the nerve theorem, to finish the proof of Theorem 1 it suffices to show that, a.a.s, **a)** $\bigcup_i \Delta_i = \mathcal{R}(n, r)$, **b)** every nonempty finite intersection $\bigcap_{i \in I} \Delta_i$ is contractible, and **c)** $\bigcap_{i \in I} A_i \neq \emptyset$ if and only if $\bigcap_{i \in I} \Delta_i \neq \emptyset$. (Note that **c)** in fact states that the nerves $\mathcal{N}((A_i)_{i=1, \dots, N})$ and $\mathcal{N}((\Delta_i)_{i=1, \dots, N})$ are identical.)

The demand **a)** immediately follows from item **(ii)** of Lemma 9. That $\bigcap_{i \in I} \Delta_i \neq \emptyset$ implies $\bigcap_{i \in I} A_i \neq \emptyset$ is immediate from the definition of Δ_i . If $\bigcap_{i \in I} A_i \neq \emptyset$ then in fact there is a ball

$$B(x, \delta_2 (\ln n/n)^{1/d}) \subseteq B(x, \delta_4 r) \subseteq \bigcap_{i \in I} A_i.$$

As we have seen above, a.a.s., every such disk contains at least one of the random points X_1, \dots, X_n . But then this random point is a vertex of $\bigcap_{i \in I} \Delta_i$ by definition of the subcomplexes Δ_i . This establishes that **c)** holds.

It remains to see why **b)** holds. Suppose that $\bigcap_{i \in I} \Delta_i$ is nonempty, and let X_{j_1}, \dots, X_{j_k} denote its vertices. We can assume without loss of generality that the points are labelled by non-decreasing distance to X_{j_1} , i.e.

$$\|X_{j_2} - X_{j_1}\| \leq \dots \leq \|X_{j_k} - X_{j_1}\|.$$

By item **(v)** of Lemma 9, for each $2 \leq \ell \leq k$, we have either $\|X_{j_\ell} - X_{j_1}\| < r$ or there is a ball $B(z, \delta_4 r) \subseteq W(X_{j_\ell}, X_{j_1}, r) \cap \bigcap_{i \in I} A_i$. By the choice of c , a.a.s., any such ball contains one of the random points X_1, \dots, X_n . In particular, $B(z, \delta_4 r)$ contains one of $X_{j_1}, \dots, X_{j_{\ell-1}}$. In other words, for each $2 \leq \ell \leq k$, we have either $\|X_{j_\ell} - X_{j_1}\| < r$ or there exists a $2 \leq m(\ell) < \ell$ such that $X_{j_{m(\ell)}} \in W(X_{j_\ell}, X_{j_1}, r)$. Writing G_ℓ for the subgraph of the random geometric graph induced by $X_{j_1}, \dots, X_{j_\ell}$, we see that

$$N_{G_\ell}[X_{j_\ell}] \subseteq N_{G_\ell}[X_{j_{m(\ell)}}],$$

where we set $m(\ell) := 1$ when $\|X_{j_\ell} - X_{j_1}\| < r$. By repeated applications of Lemma 5 we now derive

$$\bigcap_{i \in I} \Delta_i = \mathcal{K}(G_k) \simeq \mathcal{K}(G_{k-1}) \simeq \cdots \simeq \mathcal{K}(G_1).$$

So $\bigcap_{i \in I} \Delta_i$ is indeed contractible. ■

3.2 The proof of Theorem 2

Much of the proof of Theorem 1 can be reused. The next two lemmas represent the main adaptations that need to be made.

Lemma 10 *Let $K \subseteq \mathbb{R}^d$ be a compact, smooth d -manifold with boundary. There exist constants $\delta_3 = \delta_3(K), \rho_1 = \rho_1(K) > 0$ such that for every $x \in K$ and $0 < r \leq \rho_1$ there is a ball $B(z, \delta_3 r) \subseteq B(x, r) \cap K$.*

Lemma 11 *Let $K \subseteq \mathbb{R}^d$ be a compact, smooth d -manifold with boundary. For every $\lambda > 0$ there exists a $\rho_2 = \rho_2(K, \lambda) > 0$ such that if $x \in K$ and $B(y, r) \subseteq K$ and $\|x - y\| \leq \lambda r$ for some $0 < r < \rho_2$ then $\text{conv}(\{x, y\}) \subseteq K$.*

Before proceeding with the proof of these two lemmas, we first explain how they can be used to adapt the proof of Theorem 1 to obtain a proof of Theorem 2. We observe that the proof of Lemma 7 carries over verbatim if we substitute the use of Lemma 8 with Lemma 10.

Corollary 12 *Suppose $K \subseteq \mathbb{R}^d$ is a compact, smooth d -manifold with boundary. There is a $\delta_2 = \delta_2(K, \nu)$ such that if $r_n \geq \delta_2 \cdot \left(\frac{\ln n}{n}\right)^{1/d}$ then, a.a.s., $K \subseteq \bigcup_{i=1}^n B(X_i, r_n)$.*

With some minor changes to the statement and its proof, we obtain the following variant of Lemma 9.

Lemma 13 *Let $K \subseteq \mathbb{R}^d$ be a compact, smooth d -manifold with boundary. There exist constants $\delta_4 = \delta_4(K), \delta_5 = \delta_5(K), \rho_3 = \rho_3(K)$ such that, for every $0 < r \leq \rho_3$, there exists a finite family of sets $A_1, \dots, A_N \subseteq K$ such that*

- (i) *Each A_i is open in the relative topology of K , and;*
- (ii) *For every $S \subseteq K$ with $\text{diam}(S) \leq r$ there is an $1 \leq i \leq N$ such that $S \subseteq A_i$, and;*
- (iii) *For every $I \subseteq [N]$, if $\bigcap_{i \in I} A_i$ is nonempty then it is contractible, and;*
- (iv) *For every $I \subseteq [N]$, if $\bigcap_{i \in I} A_i$ is nonempty then it contains a ball of radius $\delta_4 \cdot r$, and;*
- (v) *For every $I \subseteq [N]$, if $x, y \in \bigcap_{i \in I} A_i$ satisfy $\|x - y\| \geq r$ and $B(y, (\delta_4/2) \cdot r) \subseteq \bigcap_{i \in I} A_i$ then $W(x, y, r) \cap \left(\bigcap_{i \in I} A_i\right)$ contains a ball of radius $\delta_5 \cdot r$.*

Proof of Lemma 13 assuming Lemmas 10 and 11. The proof is largely the same as the proof of Lemma 9. The construction of A_1, \dots, A_N carries over unaltered, as do the proofs of (i), (ii) and (iv) except that we substitute the use of Lemma 8 with Lemma 10 and have to assume the corresponding upper bounds on r . For part (iii) we can no longer

use convexity. Instead, we rely on Lemma 11, as follows. Suppose $\bigcap_{i \in I} A_i$ is non-empty. We have already established that there is a ball $B(x, \delta_4 \cdot r) \subseteq \bigcap_{i \in I} A_i$. Since $\text{diam}(\bigcap_{i \in I} A_i) \leq 8r$, under the assumption that $r \leq \rho_2(K, 8/\delta_4)$, we know that for all $y \in \bigcap_{i \in I} A_i$ the line segment $\text{conv}(\{x, y\})$ is contained in K . Since, for each i , we have defined $A_i := B(x_i, s_i) \cap K$ and $B(x_i, s_i)$ is convex, it follows that

$$\text{conv}(\{x, y\}) \subseteq \bigcap_{i \in I} A_i.$$

In other words, for each $y \in \bigcap_{i \in I} A_i$, the line segment between y and x is completely contained in $\bigcap_{i \in I} A_i$. So $\bigcap_{i \in I} A_i$ is star-shaped, and in particular contractible.

To see that **(v)** holds, we suppose that $B(y, (\delta_4/2) \cdot r) \subseteq \bigcap_{i \in I} A_i$ and $x \in \bigcap_{i \in I} A_i$. For each $z \in B(y, (\delta_4/4) \cdot r)$ we of course have

$$B(z, (\delta_4/4) \cdot r) \subseteq B(y, (\delta_4/2) \cdot r) \subseteq K.$$

Therefore, applying Lemma 11, provided $r \leq \rho_2(K, 16/\delta_4)$, we have that $\text{conv}(\{z, x\}) \subseteq K$ for all $z \in B(y, (\delta_4/4) \cdot r)$. As $A_i = B(x_i, s_i) \cap K$ and balls are convex, it follows that in fact $\text{conv}(\{z, x\}) \subseteq \bigcap_{i \in I} A_i$. Hence, also

$$X := \text{conv}(B(y, (\delta_4/4) \cdot r) \cup \{x\}) \subseteq \bigcap_{i \in I} A_i.$$

Applying Lemma 6, there is a $z \in \text{conv}(\{x, y\})$ such that $B(z, \delta_1 r) \subseteq W(x, y, r)$ where $\delta_1 = \delta_1(8)$. We now apply Lemma 8 to X . Note that $\text{diam}(X) \leq 8r$, $\text{inr}(X) \geq \delta_4/4$. Hence, taking $\delta_3 = \delta_3(32/\delta_4)$ as provided by Lemma 8, we have

$$B(z, \delta_3 \delta_1 r) \subseteq X \cap W(x, y, r) \subseteq \left(\bigcap_{i \in I} A_i \right) \cap W(x, y, r).$$

In other words, we have established **(v)** for $\delta_5 := \delta_1 \delta_3$. ■

Proof of Theorem 2 assuming Lemmas 10 and 11. The proof is largely the same as the proof of Theorem 1. We assume $c_1 \cdot (\ln n/n)^{1/d} \leq r \leq c_2$ where the choice of the constants will be determined in the course of the proof. Having chosen c_2 appropriately we can apply Lemma 13 to obtain A_1, \dots, A_N . We again let Δ_i denote subcomplex of $\mathcal{R}(n, r)$ induced by the points $\{X_1, \dots, X_n\} \cap A_i$. It again suffices to show that, a.a.s, **a)** $\bigcup_i \Delta_i = \mathcal{R}(n, r)$, **b)** every nonempty finite intersection $\bigcap_{i \in I} \Delta_i$ is contractible, and **c)** $\bigcap_{i \in I} A_i \neq \emptyset$ if and only if $\bigcap_{i \in I} \Delta_i \neq \emptyset$.

The demand **a)** immediately follows from item **(ii)** of Lemma 13, and that **c)** holds a.a.s. follows in the same way as in the proof of Theorem 1 (assuming the constant c_1 was chosen sufficiently large) except that we use Corollary 12 in place of Lemma 7.

The proof of **b)** needs slightly more adaptation. Suppose again that for some $I \subseteq [N]$ the intersection $\bigcap_{i \in I} \Delta_i$ is nonempty and let X_{j_1}, \dots, X_{j_k} denote its vertices. Since $\bigcap_{i \in I} A_i \supseteq B(z, \delta_4 r)$ for some z , having chosen c_1 appropriately large, we can assume without loss of generality that $X_{j_1} \in B(z, (\delta_4/2)r)$. We also assume, without loss of generality, that the points X_{j_1}, \dots, X_{j_k} are labelled by non-decreasing distance to X_{j_1} .

That

$$X_{j_1} \in B(z, (\delta_4/2)r) \subseteq B(z, \delta_4 r) \subseteq \bigcap_{i \in I} A_i,$$

of course implies that

$$B(X_{j_1}, (\delta_4/2)r) \subseteq \bigcap_{i \in I} A_i.$$

Applying part **(v)** of Lemma 13, for each $2 \leq \ell \leq k$ we have that either $\|X_{j_\ell} - X_{j_1}\| \leq r$ or $W(X_{j_\ell}, X_{j_1}, r) \cap \bigcap_{i \in I} A_i$ contains a ball of radius $\delta_5 r$. Having chosen c_1 sufficiently large, a.a.s., any such ball contains one of the random points X_1, \dots, X_n . In other words, for each $2 \leq \ell \leq k$, we have either $\|X_{j_\ell} - X_{j_1}\| < r$ or there is an $2 \leq m(\ell) < \ell$ such that $X_{j_{m(\ell)}} \in W(X_{j_\ell}, X_{j_1}, r)$. We can thus conclude **(b)** holds in the same way we did in the proof of Theorem 1. \blacksquare

It remains to prove Lemmas 10 and 11. That K is a smooth d -manifold with boundary means that for every $x \in K$ there are open sets $O, U \subseteq \mathbb{R}^d$ with $x \in O$ and a diffeomorphism (a smooth bijection whose inverse is also smooth) $\varphi : O \rightarrow U$ such that $\varphi[O \cap K] = U \cap \mathbb{H}^d$, where $\mathbb{H}^d := [0, \infty) \times \mathbb{R}^{d-1}$. If K is also compact we have in addition:

Lemma 14 *Let $K \subseteq \mathbb{R}^d$ be a compact, smooth d -manifold with boundary. There exist $\rho_4 = \rho_4(K) > 0$ and $\delta_6 = \delta_6(K)$ such that, for every $x \in K$ there exist open sets $O_x, U_x \subseteq \mathbb{R}^d$ and a diffeomorphism $\varphi_x : O_x \rightarrow U_x$, such that*

- (i) $\varphi_x[O_x \cap K] = U_x \cap \mathbb{H}^d$, $B(x, \rho_4) \subseteq O_x$, $B(\varphi_x(x), \rho_4) \subseteq U_x$, and;
- (ii) *The absolute values of the partial derivatives of φ_x of order one and two are bounded by δ_6 on O_x , and;*
- (iii) *The absolute values of the partial derivatives of φ_x^{-1} of order one and two are bounded by δ_6 on U_x .*

Proof. For each $x \in K$ there are open sets $V_x, W_x \subseteq \mathbb{R}^d$ with $x \in V_x$ and a diffeomorphism $\psi_x : V_x \rightarrow W_x$ such that $\psi_x[V_x \cap K] = W_x \cap \mathbb{H}^d$. We can assume without loss of generality that, for each $x \in K$, there exists a $c_x < \infty$ such that all partial derivatives of ψ_x, ψ_x^{-1} of orders up to two are bounded in absolute value by c_x . (Switching to a smaller open subset $V'_x \subseteq V_x$ if needed.)

For each $x \in K$ there exists an $s_x > 0$ such that $B(x, s_x) \subseteq V_x$ and $B(\psi_x(x), s_x) \subseteq W_x$. Let us write

$$A_x := B(x, s_x/2) \cap \psi_x^{-1}[B(\psi_x(x), s_x/2)].$$

Since the sets A_x are open and K is compact, there are x_1, \dots, x_N such that $K \subseteq \bigcup_{i=1}^N A_{x_i}$. We set

$$\rho_4 := \frac{1}{2} \cdot \left(\min_{i=1, \dots, N} s_{x_i} \right), \quad c := \max_{i=1, \dots, N} c_{x_i},$$

and for each $x \in K$ we fix an i such that $x \in A_{x_i}$ and set:

$$O_x := V_{x_i}, \quad U_x := W_{x_i}, \quad \varphi_x := \psi_{x_i}.$$

Observe that $x \in A_{x_i}$ implies that

$$B(x, \rho_4) \subseteq B(x_i, 2\rho_4) \subseteq B(x_i, s_x) \subseteq V_{x_i} = O_x,$$

and $\psi_{x_i}(x) \in B(\psi_{x_i}(x_i), s_{x_i}/2)$ so that also

$$B(\varphi_x(x), \rho_4) = B(\psi_{x_i}(x), \rho_4) \subseteq B(\psi_{x_i}(x_i), 2\rho_4) \subseteq B(\psi_{x_i}(x_i), s_{x_i}) \subseteq W_{x_i} = U_x.$$

■

Corollary 15 *Let $K \subseteq \mathbb{R}^d$ be a compact, smooth d -manifold with boundary. There exist $\rho_5 = \rho_5(K), \delta_7 = \delta_7(K) > 0$ such that, for every $x \in K$ and $0 < r < \rho_5$:*

- (i) *For every $y \in B(x, \rho_5)$ we have $B(\varphi_x(y), \delta_7 r) \subseteq \varphi_x[B(y, r)]$, and;*
- (ii) *For every $y \in B(\varphi_x(x), \rho_5)$ we have $B(\varphi_x^{-1}(y), \delta_7 r) \subseteq \varphi_x^{-1}[B(y, r)]$.*

where φ_x is as in Lemma 14.

Proof. It follows from Lemma 14, using for instance Theorem 9.19 of [26], that for every $x \in K$ and $y, z \in B(x, r_0)$ and $y', z' \in B(\varphi_x(x), r_0)$ we have

$$\|\varphi_x(y) - \varphi_x(z)\| \leq \delta_6 \sqrt{d} \cdot \|y - z\|, \quad \|y' - z'\| \leq \delta_6 \sqrt{d} \cdot \|\varphi_x^{-1}(y') - \varphi_x^{-1}(z')\|.$$

Hence

$$\varphi_x^{-1}[B(\varphi_x(y), \delta_7 r)] \subseteq B(\varphi_x^{-1}(\varphi_x(y)), \delta_6 \sqrt{d} \cdot \delta_7 r) = B(y, \delta_6 \sqrt{d} \cdot \delta_7 r) \subseteq B(y, r),$$

the first inclusion holding provided $r, \|y - x\| < \rho_5 := \rho_4/2$ and the last inclusion holding provided $\delta_7 < \min(1/\delta_6 \sqrt{d}, 1)$. In other words, with this choice of δ_7, ρ_5 we have:

$$B(\varphi_x(y), \delta_7 r) \subseteq \varphi_x[B(y, r)],$$

establishing (i). The proof of (ii) is completely analogous. ■

Proof of Lemma 10. Let $x \in K$ be arbitrary and $r < \rho_5(K)$ with ρ_5 as provided by Corollary 15. Applying Corollary 15, we have

$$\varphi_x[B(x, r)] \supseteq B(\varphi_x(x), \delta_7 r).$$

Since $\varphi_x(x) \in \mathbb{H}^d$, there is a ball

$$B(z, (\delta_7/2) \cdot r) \subseteq B(\varphi_x(x), \delta_7 r) \cap \mathbb{H}^d,$$

of radius $(\delta_7/2) \cdot r$ contained in $B(\varphi_x(x), \delta_7 r) \cap \mathbb{H}^d$. Applying Corollary 15 once again, we have

$$B(x, r) \cap K \supseteq \varphi_x^{-1}[B(z, (\delta_7/2) \cdot r)] \supseteq B(\varphi_x^{-1}(z), (\delta_7^2/2) \cdot r).$$

We can conclude Lemma 10 holds with $\delta_3 := \delta_7^2/2$ and $\rho_1 := \rho_5$. ■

Proof of Lemma 11. We assume $x \in K, B(y, r) \subseteq K$ and $\|x - y\| \leq \lambda r$ with $0 < r \leq \rho_2$ where the constant ρ_2 will be determined in the course of the proof. We need to show that for every $t \in [0, 1]$ the point $(1 - t)x + ty \in K$. Or, equivalently, $\varphi_x((1 - t)x + ty) \in \mathbb{H}^d$ with φ_x as provided by Lemma 14 (assuming we have chosen ρ_2 so that $\lambda\rho_2 < \rho_4$.) For notational convenience we will write

$$\psi(t) := \varphi_x((1 - t)x + ty), \quad \hat{\psi}(t) := \varphi_x(x) + (D\varphi_x)_x \cdot t(y - x),$$

where $(D\varphi_x)_z$ denotes the matrix of first derivatives of φ_x evaluated at z . By the multivariate Taylor theorem:

$$\|\psi(t) - \hat{\psi}(t)\| \leq \alpha \cdot \delta_6 \cdot t^2 \cdot \|y - x\|^2 \leq \alpha \cdot \delta_6 \cdot \lambda^2 \cdot t^2 \cdot r^2, \quad (3)$$

where δ_6 is as given by Lemma 14 and $\alpha = \alpha(d)$ is a constant that depends only on the dimension d . By Corollary 15 we have

$$B(\psi(1), \delta_7 r) = B(\varphi_x(y), \delta_7 r) \subseteq \varphi_x[B(y, r)] \subseteq \mathbb{H}^d.$$

Assuming $r < \delta_7/(2\alpha\delta_6\lambda^2)$, we have $\|\psi(1) - \hat{\psi}(1)\| < (\delta_7/2) \cdot r$ and hence

$$B(\hat{\psi}(1), (\delta_7/2) \cdot r) \subseteq \mathbb{H}^d.$$

Since also $\psi(0) = \varphi_x(x) \in \mathbb{H}^d$, we have

$$\text{conv}(\{\psi(0)\} \cup B(\hat{\psi}(1), (\delta_7/2) \cdot r)) \subseteq \mathbb{H}^d.$$

In particular

$$\{(1 - t)\psi(0) + tz : z \in B(\hat{\psi}(1), (\delta_7/2) \cdot r)\} = B(\hat{\psi}(t), (\delta_7/2) \cdot tr) \subseteq \mathbb{H}^d.$$

By (3) and $r \leq \rho_2$, having chosen the constant ρ_2 sufficiently small, we have $\psi(t) \in B(\hat{\psi}(t), (\delta_7/2) \cdot tr)$. In other words, we have established that $\varphi((1 - t)x + ty) \in \mathbb{H}^d$ as required, for a suitable choice of the constant $\rho_2(K, \lambda)$. ■

4 Discussion

4.1 A connection with the game of cops and robbers

The game of cops and robbers is a two-player game played on a graph. There are two players, a cop and a robber, each located on one of the vertices of the graph. Before the game begins, the cop chooses his starting vertex and after that the robber chooses his (he is allowed to take into account where the cop is when choosing his starting position). In each turn, the cop either stays put or moves to a vertex adjacent to a vertex he is currently on. After the cop has made his move the robber does the same, and the next turn starts. The goal of the cop is to capture the robber, i.e. to be located on the same vertex as the robber. If a graph G is such that the cop has a strategy so that no matter how the robber plays, the cop is guaranteed to achieve his goal in a finite number of moves, then G is cop-win. Aigner and Fromme [2], and independently Quilliot [25], have shown that a graph is cop-win if and only if it can be reduced to a single vertex via a sequence of deletions as in Lemma 5. Thus:

Corollary 16 *If G is cop-win then $\mathcal{K}(G)$ is contractible.*

In particular, $\mathcal{R}(n, r)$ is contractible when the corresponding random geometric graph (its 1-skeleton) is cop-win. It is however known [5] that the random geometric graph in two dimensions is not cop-win a.a.s. for all $r \leq c \ln n / \sqrt{n}$, for some constant $c > 0$. (The result in [5] is stated only for the uniform distribution on the unit square, but the proof easily adapts with very little modification to the current setting restricted to two dimensions.) Note that the stated bound is a multiplicative factor $\Omega(\sqrt{\ln n})$ larger than the bound $O(\sqrt{\ln n/n})$ we have established in Theorem 1 for contractibility in two dimensions. Our proofs did however make use of ideas from [5] that were used to show the random geometric graph is a cop-win for $r = \Omega((\ln n/n)^{1/5})$ in two dimensions. (An alternative proof was given by Alon and Pralat [3].) Part of our proof was to show that the random geometric graph is “locally cop-win” when $r \geq \text{const} \cdot (\ln n/n)^{1/d}$.

4.2 Suggestions for further work

In the light of several results on random geometric graphs (e.g. [4, 23, 24]) and random Čech complexes (e.g. [10]) in the regime when $r = \Theta((\ln n/n)^{1/d})$ it seems natural to expect a “sharp threshold” for contractibility.

Conjecture 17 *If $K \subseteq \mathbb{R}^d$ is a convex body with smooth boundary and the distribution is uniform on K then there exists $c = c(K)$ such that, for every fixed $\varepsilon > 0$:*

$$\mathbb{P}(\mathcal{R}(n, r_n) \text{ is contractible}) \xrightarrow{n \rightarrow \infty} \begin{cases} 0 & \text{if } r_n \leq (c - \varepsilon) \left(\frac{\ln n}{n}\right)^{1/d}, \\ 1 & \text{if } r_n \geq (c + \varepsilon) \left(\frac{\ln n}{n}\right)^{1/d}. \end{cases}$$

In fact we would expect this to hold without the assumption that the boundary of K is smooth and under more general assumptions on the probability measure, but we have opted for this version to increase the chances of success for whoever chooses to attempt the problem.

Having another look at the proof of Lemma 5, we see that it is in fact shown that $\mathcal{K}(G \setminus v)$ is a strong deformation retract of $\mathcal{K}(G)$ if the closed neighbourhood of v is contained in the closed neighbourhood of some other vertex. What is more, one can check that the operation of removing v can be described as a sequence of collapses. So in fact when G is cop-win then $\mathcal{K}(G)$ is *collapsible*. (The definitions of strong deformation retract and collapse can for instance be found in Section 6.4 of [16].) As mentioned in the previous section, there is a choice of $r_n \gg \left(\frac{\ln n}{n}\right)^{1/d}$ such that the random geometric graph $G(n, r_n)$ a.a.s. is not cop-win. This leads us to the following conjecture.

Conjecture 18 *Suppose $K \subseteq \mathbb{R}^d$ is a convex body with smooth boundary and the distribution is uniform on K . There exists a sequence $\rho_n \gg \left(\frac{\ln n}{n}\right)^{1/d}$ such that $\mathcal{R}(n, r_n)$ is a.a.s. not collapsible whenever $r_n \leq \rho_n$.*

Another natural direction for further research would be to consider a setup where the random points are not all contained in K , but there is a small amount of random noise added to the point locations. That is, we initially choose each point randomly from a compact, smooth d -manifold with boundary but then we add a small i.i.d. “error” to it. It would be interesting to determine under which conditions on the noise our Theorems 1 and 2 are still

valid. Previous work in this spirit includes [22] and [9]. Of course when the noise comes from a fixed distribution with unbounded support (e.g. a multivariate standard normal) then there is no chance of direct analogues of Theorems 1 and 2 being valid due to “outliers” (see [1]). Instead, it makes sense to consider the fairly natural case where the co-variance matrix of the noise decays as a function of n , and characterize the behaviour in terms of the rate of decay, perhaps obtaining a kind of threshold result.

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Data availability statement

Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

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