The number of disk graphs

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Abstract

A disk graph is the intersection graph of disks in the plane, and a unit disk graph is the intersection graph of unit radius disks in the plane. We give upper and lower bounds on the number of labelled unit disk and disk graphs on n vertices. We show that the number of unit disk graphs on n vertices is $n^{2n} \cdot \alpha(n)^n$ and the number of disk graphs on n vertices is $n^{3n} \cdot \beta(n)^n$, where $\alpha(n)$ and $\beta(n)$ are $\Theta(1)$. We conjecture that there exist constants α, β such that the number of unit disk graphs is $n^{2n} \cdot (\alpha + o(1))^n$ and the number of disk graphs is $n^{3n} \cdot (\beta + o(1))^n$.

1 Introduction and statement of results

A *disk graph* is the intersection graph of open disks in the plane; that is, we can represent each vertex i by a disk D_i in such a way that $ij \in E(G)$ if and only if $D_i \cap D_j \neq \emptyset$ If we can take $D_1, \ldots, D_n$ all of equal radius, then G is a *unit disk graph*.

Partly because of their relevance for practical applications (for example in cellular communication systems) disk graphs have been the subject of a sustained research effort by many different authors over the past 20 years or so. Aspects of these graphs that have been studied include the algorithmic decision problem of "recognizing" (unit) disk graphs [5, 12], computing or approximating the chromatic number [6, 10, 15], computing the clique number [6, 20], computing or approximating the independence number [6, 7, 13, 15, 18] and finding or approximating a smallest dominating set [6, 17]. Rather surpisingly, it appears that the question of how many of the $2^{\binom{n}{2}}$ labelled graphs on n vertices are (unit) disk graphs has largely escaped attention.

Let us denote by $\mathcal{UDG}(n)$ the number of labelled graphs on n vertices that are (isomorphic to) unit disk graphs, and let $\mathcal{DG}(n)$ denote the number of disk graphs on n vertices. We will prove the following two results.

Theorem 1.1 *There exist constants $\alpha, \beta > 0$ such that*

$$n^{2n}\alpha^n \leq \mathcal{UDG}(n) \leq n^{2n}\beta^n.$$

Theorem 1.2 *There exist constants $\alpha, \beta > 0$ such that*

$$n^{3n}\alpha^n \leq \mathcal{DG}(n) \leq n^{3n}\beta^n.$$

The upper bounds in these theorems have similar short proofs, applying a result of Warren [23] (stated as Theorem 3.1 below) that gives an upper bound on the number of distinct "sign patterns" a system of polynomials can achieve. Warren's theorem is in fact a refinement of an earlier result proved independently by Oleĭnik and Petrovskiĭ [19], Milnor [16] and Thom [22]. The method of upper bounding the size of some class of combinatorial objects by applying either Warren's theorem or the Milnor-Oleĭnik-Petrovskiĭ-Thom theorem has previously been applied several times by others in the context of enumerating combinatorial objects, for instance by Goodman and Pollack [9]

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and Alon [2] to bound the number of combinatorial types of polytopes or point configurations; and by Alon and Scheinerman [4] to bound the number of containment posets. See for example chapter 6 of the book *Lectures on Discrete Geometry* by Matoušek [14] for further background. Spinrad [21] has previously pointed out that Warren's theorem can be used to bound the size of various geometrically defined graph classes including disk graphs. In section 3 we present the short argument deriving the upper bounds for completeness.

What is unusual in the case of (unit) disk graphs is that we are able to give explicit lower bound constructions that are "only" an exponential factor away from the upper bound given by Warren's theorem. Most of the bulk of the present paper is taken up by our lower bound proofs. Let us also remark that with less effort weaker lower bounds of the form $\mathcal{UDG}(n) \geq n^{(2-o(1))n}$ and $\mathcal{DG}(n) \geq n^{(3-o(1))n}$ can be achieved. In Section 6 we derive bounds on the number of unit disk graphs in the case when the maximum clique size is bounded by some function of n. In Section 7 we briefly discuss dimensions other than 2, and in particular consider unit interval graphs.

To conclude the introduction, we offer two natural conjectures.

Conjecture 1.3 *There exists a constant $\alpha > 0$ such that*

$$\mathcal{UDG}(n) = n^{2n} \cdot (\alpha + o(1))^n.$$

Conjecture 1.4 *There exists a constant $\beta > 0$ such that*

$$\mathcal{DG}(n) = n^{3n} \cdot (\beta + o(1))^n.$$

2 Notation and preliminaries

We shall use the convention that $[n] := \{1, \ldots, n\}$. If A is a set then $\binom{A}{k}$ denotes the collection of all subsets of A of cardinality exactly k. If $A_1, \ldots, A_n$ are sets then $G(A_1, \ldots, A_n)$ will denote their intersection graph. That is, the graph $G = (V, E)$ with vertex set $V = [n]$ and an edge $ij \in E$ if and only if $A_i \cap A_j \neq \emptyset$.

We will make use of the following lower bound for the factorial

$$k! \geq \left(\frac{k}{e}\right)^k \qquad \text{for all } k \geq 1. \tag{1}$$

This can for instance be seen from the precise bounds on Stirling's approximation in [8], section 11.9.

All disks in this paper will be open unless explicitly stated otherwise. We shall denote the (open) disk with center p and radius $r > 0$ by:

$$B(p, r) := \{q \in \mathbb{R}^2 : \|p - q\| < r\}.$$

The circle of center p and radius $r > 0$ will be denoted by:

$$S(p, r) := \partial B(p, r) = \{q \in \mathbb{R}^2 : \|p - q\| = r\}.$$

If $D \subseteq \mathbb{R}^2$ is a disk then we shall denote its center by $p(D)$ and its radius by $r(D)$.

We will use the following standard elementary lemma. We leave the straightforward computations verifying the lemma to the reader.

Lemma 2.1 *Let $p_1, p_2 \in \mathbb{R}^2$ and $r_1, r_2 > 0$ be such that, setting $d := \|p_1 - p_2\|$, we have $r_1 + r_2 > d$ and $r_1 < d + r_2$ and $r_2 < d + r_1$. Then the two intersection points $\{q_1, q_2\} = S(p_1, r_1) \cap S(p_2, r_2)$ of the circles $S(p_1, r_1), S(p_2, r_2)$ can be expressed as:*

$$q_1 = p_1 + \left(\frac{d^2-r_2^2+r_1^2}{2d^2}\right)(p_2-p_1) - \sqrt{\frac{4d^2r_1^2-(d^2-r_2^2+r_1^2)^2}{4d^4}}\begin{pmatrix} (p_1)_y-(p_2)_y \\ (p_2)_x-(p_1)_x \end{pmatrix},$$

$$q_2 = p_1 + \left(\frac{d^2-r_2^2+r_1^2}{2d^2}\right)(p_2-p_1) + \sqrt{\frac{4d^2r_1^2-(d^2-r_2^2+r_1^2)^2}{4d^4}}\begin{pmatrix} (p_1)_y-(p_2)_y \\ (p_2)_x-(p_1)_x \end{pmatrix}.$$

We will say that two disks D_1, D_2 *kiss* if their closures intersect but their interiors are disjoint. In other words they are tangent and do not overlap.

In section 5 we will use the following classical result. This, or something like it, was obtained by Princess Elizabeth of Bohemia in a letter to Descartes in 1643, but her letter does not survive (see [11]). We have not been able to find a write-up anywhere in the literature and we have therefore resorted to writing up a solution ourselves for completeness, to be found in Appendix 7.

Theorem 2.2 (Elizabeth of Bohemia, 1643) *Let D_1, D_2, D_3 be three distinct disks in the plane, such that none of them is contained in the union of the other two. Set $p_i := p(D_i), r_i := r(D_i)$ and denote $d_{ij} := \|p_i - p_j\|$. The following hold:*

(i) *If there exists a disk kissing D_1, D_2, D_3 then its radius s satisfies the quadratic equation:*

$$a(s+r_1)^2 + b(s+r_1) + c = 0, \tag{2}$$

where

$$\begin{aligned} a = \ & \left(\frac{r_1-r_2}{d_{12}}\right)^2 + \left(\frac{r_1-r_3}{d_{13}}\right)^2 - \left(\frac{d_{12}^2+d_{13}^2-d_{23}^2}{d_{12}d_{13}}\right)\left(\frac{r_1-r_2}{d_{12}}\right)\left(\frac{r_1-r_3}{d_{13}}\right) \\ & + \left(\frac{d_{12}^2+d_{13}^2-d_{23}^2}{2d_{12}d_{13}}\right)^2 - 1, \\ b = \ & 2\left(\frac{r_1-r_2}{d_{12}}\right)\left(\frac{d_{12}^2-(r_1-r_2)^2}{2d_{12}}\right) + 2\left(\frac{r_1-r_3}{d_{13}}\right)\left(\frac{d_{13}^2-(r_1-r_3)^2}{2d_{13}}\right) \\ & - \left(\frac{d_{12}^2+d_{13}^2-d_{23}^2}{d_{12}d_{13}}\right)\left[\left(\frac{r_1-r_2}{d_{12}}\right)\left(\frac{d_{13}^2-(r_1-r_3)^2}{2d_{13}}\right) + \left(\frac{r_1-r_3}{d_{13}}\right)\left(\frac{d_{12}^2-(r_1-r_2)^2}{2d_{12}}\right)\right], \\ c = \ & \left(\frac{d_{12}^2-(r_1-r_2)^2}{2d_{12}}\right)^2 + \left(\frac{d_{13}^2-(r_1-r_3)^2}{2d_{13}}\right)^2 \\ & - \left(\frac{d_{12}^2+d_{13}^2-d_{23}^2}{d_{12}d_{13}}\right)\left(\frac{d_{12}^2-(r_1-r_2)^2}{2d_{12}}\right)\left(\frac{d_{13}^2-(r_1-r_3)^2}{2d_{13}}\right). \end{aligned}$$

(ii) *If p_1, p_2, p_3 are collinear and $s > 0$ is a solution of (2) then there are precisely two distinct disks kissing D_1, D_2, D_3, each of radius s, and s is the only solution of (2).*

(iii) *If p_1, p_2, p_3 are not collinear and $s > 0$ is a solution of (2) then there is precisely one disk D of radius s kissing D_1, D_2, D_3.*

3 The upper bounds

The proofs of the upper bounds are very similar, so we shall only spell out the proof for disk graphs. A graph G is a disk graph if and only if we can find $(x_1, y_1, r_1, \dots, x_n, y_n, r_n) \in \mathbb{R}^{3n}$ such that

$$\begin{array}{ll} (x_i-x_j)^2 + (y_i-y_j)^2 < (r_i+r_j)^2, & \text{for all } ij \in E(G), \\ (x_i-x_j)^2 + (y_i-y_j)^2 \geq (r_i+r_j)^2, & \text{for all } ij \notin E(G), \\ r_i > 0 & \text{for all } i \in V(G). \end{array} \tag{3}$$

Such a vector $(x_1, y_1, r_1, \dots, x_n, y_n, r_n)$ is called a representation of G. Let us observe that every disk graph has a representation in which all inequalities of (3) are strict. (Starting from an arbitrary representation, we fix the r_is and multiply the x_is and y_is by a scalar $\lambda > 1$ but sufficiently close to 1.) We shall need to following result:

Theorem 3.1 ([23]) *If $Q_1, \dots, Q_m$ are polynomials of degree at most d in real variables $z_1, \dots, z_k$ then the number of distinct "sign patterns"*

$$(\mathrm{sign}(Q_1(\overline{z})), \dots, \mathrm{sign}(Q_m(\overline{z}))) \in \{-,+\}^m$$

that occur on $\mathbb{R}^k \setminus \bigcup_{i=1}^m \{\overline{z} : q_i(\overline{z}) = 0\}$ does not exceed $\left(\frac{4edm}{k}\right)^k$.

Let us denote $P_{ij} := (x_i - x_j)^2 + (y_i - y_j)^2 - (r_i + r_j)^2$ for each $\{i,j\} \in \binom{[n]}{2}$. Then $\mathcal{DG}(n)$ is precisely the number of distinct sign patterns the P_{ij}s achieve on $\left(\mathbb{R}^2 \times (0,\infty)\right)^n$. Theorem 3.1 with $d = 2$ gives:

$$\mathcal{DG}(n) \leq \left(8e\binom{n}{2}/3n\right)^{3n} < n^{3n}64^n,$$

giving the required upper bound with $\beta = 4$.

4 Proof of the lower bound in Theorem 1.1

Let us put $P_1 = (-1,0), P_2 = (1,0), P_0 = (0,\sqrt{3}) \in \mathbb{R}^2$. Then P_0, P_1, P_2 are the corners of an equilateral triangle of side length 2. We will need the following observation.

Lemma 4.1 *There exist constants $\varepsilon_0 > 0$ and $C > 0$ such that for all $0 < \varepsilon < \varepsilon_0$, and all $p_1 \in B(P_1, \varepsilon), p_2 \in B(P_2, \varepsilon)$ there exists a unique point $q \in B(P_0, C\varepsilon)$ with $\|q - p_1\| = \|q - p_2\| = 2$.*

We postpone the proof of Lemma 4.1 until the end of this section and we first use it to prove the lower bound in Theorem 1.1. We shall place disks $D_1, \dots, D_n$ of radius 1 in the plane one by one, subject to the two conditions **($\mathcal{U}$-1)** and **($\mathcal{U}$-2)** below. (See figure 1 for a depiction of the construction.) Let the sequence $0 < \varepsilon_1 < \dots < \varepsilon_n$ be defined by $\varepsilon_i := \varepsilon_0 / C^{n-i}$ with ε_0, C as provided by Lemma 4.1 (we assume without loss of generality that $\varepsilon_0 < 10^{-10}$ and $C > 1$). Observe that this way we have $\varepsilon_{i+1} \leq C\varepsilon_i$ and $\varepsilon_i \leq \varepsilon_0$ for all i. Set $p_i := p(D_i)$ and let $S_i := S(p_i, 2)$

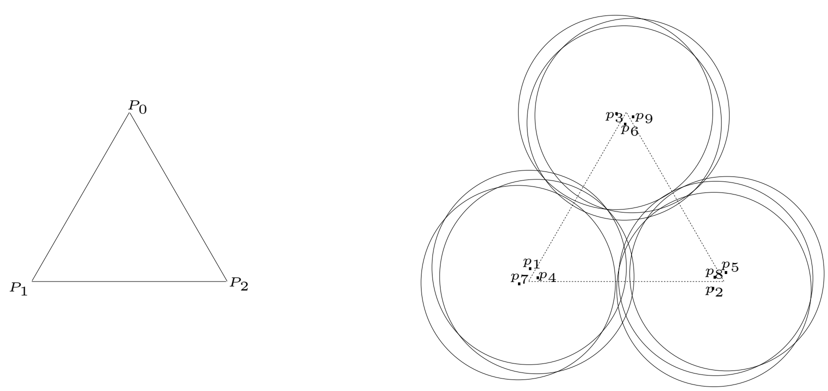

Figure 1: The construction for the UDG lower bound.

denote the circle around p_i of radius 2. We will make sure that the following always hold:

($\mathcal{U}$-1) $\|p_i - P_j\| < \varepsilon_i$ if $i \equiv j \bmod 3$.

($\mathcal{U}$-2) $S_i \cap S_j \cap S_k = \emptyset$ for all distinct i, j, k.

Observe that each pair of distinct circles S_i, S_j meet in exactly two points: condition **($\mathcal{U}$-2)** says that no point lies on more than two of the circles. For $k = 1, \dots, n$ let u_k denote the number of distinct labelled graphs $G(D_1, \dots, D_k)$ that can be represented by disks $D_1, \dots, D_k$ of radius one placed in such a way that the demands **($\mathcal{U}$-1)** and **($\mathcal{U}$-2)** are satisfied. We will show that:

$$u_{k+1} \geq \lfloor k/3 \rfloor^2 \cdot u_k \quad \text{for all } 3 \leq k \leq n-1. \tag{4}$$

Let us first briefly explain how this implies the lower bound in Theorem 1.1. Since $u_3 \geq 1$, the inequality (4) implies:

$$\begin{array}{rcl} u_n & \geq & \left(\prod_{i=3}^{n-1} \lfloor \frac{i}{3} \rfloor\right)^2 \geq \left(\prod_{i=3}^{n-1} \frac{i-2}{3}\right)^2 = \left(\frac{(n-3)!}{3^{n-3}}\right)^2 \\ & \geq & \left(\frac{n-3}{3e}\right)^{2(n-3)} \\ & \geq & \left(\frac{n}{3e}\right)^{2n} \left(\frac{n}{3}\right)^{-6}, \end{array} \tag{5}$$

where we have used (1) to get the second line; and to get the last line we used that $(1+\frac{3}{n-3})^{2(n-3)} \leq e^6$ which implies $(1-3/n)^{2(n-3)} \geq e^{-6}$. Since $(n/3)^6 < 10^n$ for all $n \geq 1$ we have

$$u_n \geq n^{2n} \alpha^n,$$

with $\alpha = 1/90e^2$. Hence, to prove the lower bound of Theorem 1.1, it indeed suffices to establish (4).

Assume that for some $3 \leq k < n$ the disks $D_1, \dots, D_k$ have been placed in a way that the demands **($\mathcal{U}$-1)** and **($\mathcal{U}$-2)** are satisfied. Let us assume $(k+1) \equiv 0 \bmod 3$. (By symmetry considerations the cases when $(k+1) \equiv 1, 2 \bmod 3$ can be dealt with analogously.)

Let us write $O := B(P_0; \varepsilon_{k+1})$. Let $\mathcal{P}$ denote the set of all pairs $\pi = (i_1, i_2) \in [k]^2$ with $(i_1, i_2) \equiv (1, 2) \bmod 3$. Then we clearly have

$$|\mathcal{P}| \geq \lfloor k/3 \rfloor^2. \tag{6}$$

For $\pi = (i_1, i_2) \in \mathcal{P}$, let p_π be defined by

$$\{p_{(i_1,i_2)}\} = S_{i_1} \cap S_{i_2} \cap O.$$

(By Lemma 4.1 p_π is well defined for all $\pi \in \mathcal{P}$. And, by **($\mathcal{U}$-2)** all the p_πs are distinct.) Let $\mathcal{N}_\pi$ be defined by

$$\mathcal{N}_{(i_1,i_2)} := B(p_{i_1}; 2) \cap B(p_{i_2}; 2).$$

Since $\varepsilon_0 < 10^{-10}$ is very small, it is easy to see (see figure 2) that:

Lemma 4.2 *For each $\pi \in \mathcal{P}$, the point p_π is the unique point of* $\mathrm{cl}(\mathcal{N}_\pi)$ *of highest y-coordinate.* ■

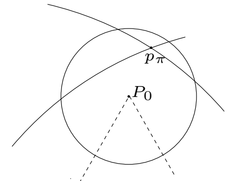

Figure 2: The point p_π is the highest point of $\mathrm{cl}(\mathcal{N}_\pi)$.

Let us now observe:

Claim 4.3 *There exist nonempty open sets $O_\pi \subseteq \mathcal{N}_\pi$, such that for all $\pi \neq \sigma \in \mathcal{P}$ we have either $O_\pi \cap \mathcal{N}_\sigma = \emptyset$ or $O_\sigma \cap \mathcal{N}_\pi = \emptyset$.*

Proof of Claim 4.3: Consider distinct $\pi, \sigma \in \mathcal{P}$. Suppose that $(p_\pi)_y \geq (p_\sigma)_y$. Then we must have that $p_\pi \in \mathrm{cl}(\mathcal{N}_\pi)$ and $p_\pi \notin \mathrm{cl}(\mathcal{N}_\sigma)$ by Lemma 4.2 and the observation that $p_\pi \neq p_\sigma$ by **($\mathcal{U}$-2)**. Thus, there is a $\delta(\pi,\sigma)$ such that $B(p_\pi, \delta(\pi,\sigma)) \cap \mathcal{N}_\sigma = \emptyset$. If $(p_\pi)_y < (p_\sigma)_y$ then we just set $\delta(\pi,\sigma) := 1$.

Let us now define, for each $\pi \in \mathcal{P}$,

$$\delta(\pi) := \min_{\substack{\sigma \in \mathcal{P}, \\ \sigma \neq \pi}} \delta(\pi, \sigma),$$

and

$$O_\pi := B(p_\pi; \delta(\pi)) \cap \mathcal{N}_\pi \cap O.$$

Then O_π is nonempty and open (note $B(p_\pi; \delta(\pi)) \cap O$ is an open neighbourhood of p_π and $p_\pi \in \mathrm{cl}(\mathcal{N}_\pi)$, so that $B(p_\pi; \delta(\pi)) \cap O$ must intersect $\mathcal{N}_\pi$.) Now let $\pi, \sigma \in \mathcal{P}$ be arbitrary but distinct. We can assume without loss of generality $(p_\pi)_y \geq (p_\sigma)_y$. Then we have $O_\pi \cap \mathcal{N}_\sigma \subseteq B(p_\pi, \delta(\pi,\sigma)) \cap \mathcal{N}_\sigma = \emptyset$. ■

For each $\pi \in \mathcal{P}$ let us now pick an arbitrary

$$q_\pi \in O_\pi \setminus \bigcup_{i=1}^{k} S_i.$$

For each $\pi \in \mathcal{P}$, let $N_\pi \subseteq [k]$ denote the set of neighbours of $k+1$ in the graph $G(D_1, \dots, D_k, B(q_\pi, 1))$.

Claim 4.4 *If $\pi \neq \sigma \in \mathcal{P}$ then $N_\pi \neq N_\sigma$.*

Proof of Claim 4.4: Let $\pi, \sigma \in \mathcal{P}$ be arbitrary but distinct. Observe that $B(q;1) \cap B(p;1) \neq \emptyset$ if and only if $q \in B(p;2)$. Thus, we have $\sigma \subseteq N_\pi$ if and only if $q_\pi \in \mathcal{N}_\sigma$. (Recall $\mathcal{N}_{(i_1,i_2)} = B(p_{i_1}, 2) \cap B(p_{i_2}, 2)$.) In particular, by Claim 4.3, $\pi \subseteq N_\pi$ and $\sigma \subseteq N_\sigma$ while either $\sigma \not\subseteq N_\pi$ or $\pi \not\subseteq N_\sigma$. ■

Combining (6) and Claim 4.4 we find that

$$u_{k+1} \geq |\mathcal{P}| \cdot u_k \geq \lfloor k/3 \rfloor^2 \cdot u_k,$$

giving (4). Thus, to prove the lower bound in Theorem 1.1 it only remains to prove Lemma 4.1.

Proof of Lemma 4.1: Let us write $p_j = P_j + (x_j, y_j)$ for $j = 1, 2$. For convenience let us set $u = (x_1, y_1, x_2, y_2)$, and let us write

$$U := [-\varepsilon_0, \varepsilon_0]^4,$$

for some sufficiently small ε_0 ($\varepsilon_0 := 10^{-10}$ will do). Using Lemma 2.1, we can express the two intersection points of $S(p_1, 2)$ and $S(p_2, 2)$ as

$$q_1(u) = \begin{pmatrix} \frac{x_1+x_2}{2} - (y_1 - y_2)\sqrt{\frac{4}{d^2} - \frac{1}{4}} \\ \frac{y_1+y_2}{2} - (2 + x_2 - x_1)\sqrt{\frac{4}{d^2} - \frac{1}{4}} \end{pmatrix},$$

$$q_2(u) = \begin{pmatrix} \frac{x_1+x_2}{2} + (y_1 - y_2)\sqrt{\frac{4}{d^2} - \frac{1}{4}} \\ \frac{y_1+y_2}{2} + (2 + x_2 - x_1)\sqrt{\frac{4}{d^2} - \frac{1}{4}} \end{pmatrix},$$

where $d = \sqrt{(2 + x_2 - x_1)^2 + (y_2 - y_1)^2}$. Clearly – having chosen ε_0 small enough - these expressions are well defined for all $u \in U$ (i.e. $0 < d < 4$ on u). What is more, all coordinates of $q_1(u), q_2(u)$ are continuously differentiable with respect to all the parameters x_1, y_1, x_2, y_2. Observe that $q_1(\underline{0}) = -P_0, q_2(\underline{0}) = P_0$. Hence – choosing ε_0 small enough – we can make sure that $q_1(u) \notin B(P_0, \varepsilon_0)$ for all $u \in U$. Since U is compact and all partial derivatives are continuous on

U, there is a constant K such that all partial derivatives are between $-K$ and K on U. Hence if $|x_1|, |y_1|, |x_2|, |y_2| < \varepsilon$ then

$$\|q_2(u) - P_0\| = \|q_2(u) - q_2(\underline{0})\| \le 4K\varepsilon,$$

so that taking $C = 4K$ proves the Lemma. ■

5 Proof of the lower bound in Theorem 1.2

The construction is very similar to the unit disk graphs one. We add a point $P_3 = (0, \frac{1}{3}\sqrt{3})$ in the middle of the equilateral triangle $P_0P_1P_2$. Let us also set $\rho_0 = \rho_1 = \rho_2 = 1$ and $\rho_3 := \frac{2}{3}\sqrt{3} - 1$.

We will again add disks $D_1, \dots, D_n$ one by one. If $i \equiv j \bmod 4$ we put a disk of radius close to ρ_j with its center close to P_j (see figure 3).

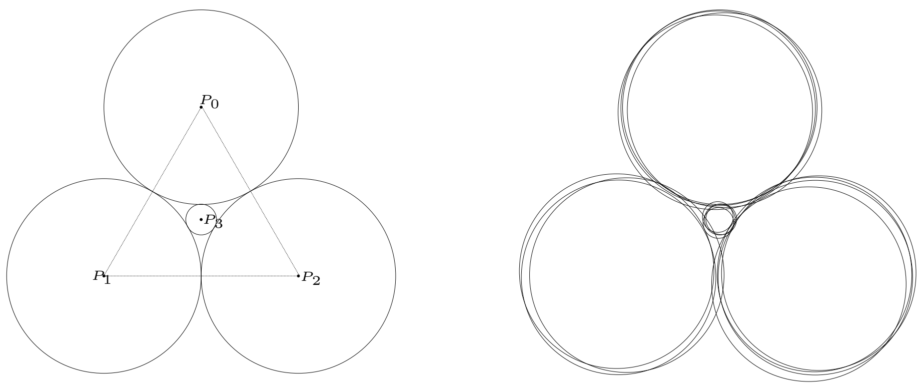

Figure 3: The construction for the disk graph lower bound.

We will use the following observation.

Lemma 5.1 *There exists constants $\varepsilon_0 > 0$ and $C > 1$ such that the following hold for all $0 < \varepsilon < \varepsilon_0$. Let D_0, D_1, D_2, D_3 be disks such that $\|p(D_j) - P_j\| < \varepsilon$ and $|r(D_j) - \rho_j| < \varepsilon$. For each $i = 0, 1, 2, 3$ there is a unique disk K_i kissing $\{D_j : j \ne i\}$. Moreover $\|p(K_i) - P_i\| < C\varepsilon$ and $|r(K_i) - \rho_i| < C\varepsilon$.*

We postpone the proof of Lemma 5.1 until the end of this section and we first use it to prove the lower bound in Theorem 1.2.

Before properly starting with the proof, let us make a few further observations. We can think of placing a new disk as choosing a point $(x, y, r) \in \mathbb{R}^3$, where $p = (x, y)$ is the center and r is the radius of the disk $D = B(p, r)$. For $D \subseteq \mathbb{R}^2$ a disk the set

$$\mathcal{C}(D) := \{(p, r) \in \mathbb{R}^3 : r > 0 \text{ and } \|p - p(D)\| < r + r(D)\},$$

corresponds exactly to all potential choices of D' such that $D' \cap D \ne \emptyset$. Observe that $\mathcal{C}(D) \subseteq \mathbb{R}^3$ is a “truncated cone” (a cone cut off by the plane $r = 0$, see figure 4).
Thus, a disk kissing D corresponds to a point on $\partial\mathcal{C}(D)$. We again postpone the proofs of the following two observations until the end of this section.

Lemma 5.2 *There is an $\varepsilon_0 > 0$ such that the following hold for all $0 < \varepsilon < \varepsilon_0$. Let $D_i = B(p_i, r_i)$ be such that $\|p_i - P_i\|, |r_i - \rho_i| < \varepsilon$ for $i = 0, 1, 2$, and let K be the disk kissing D_0, D_1, D_2 provided by Lemma 5.1. Then $(p(K), r(K))$ is the unique point of smallest r-coordinate of $\mathrm{cl}(\mathcal{C}(D_0) \cap \mathcal{C}(D_1) \cap \mathcal{C}(D_2))$.*

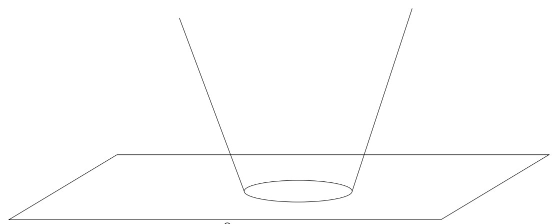

Figure 4: The "truncated cone" corresponding to all disks intersecting a given disk.

Lemma 5.3 *There is an $\varepsilon_0 > 0$ such that the following hold for all $0 < \varepsilon < \varepsilon_0$. Let $D_i = B(p_i, r_i)$ be such that $\|p_i - P_i\|, |r_i - \rho_i| < \varepsilon$ for $i = 1, 2, 3$, and let K be the disk kissing D_1, D_2, D_3 provided by Lemma 5.1. Then $(p(K), r(K))$ is the unique point of largest r-coordinate of* $\mathrm{cl}(\mathcal{C}(D_3) \setminus \mathrm{cl}(\mathcal{C}(D_1) \cup \mathcal{C}(D_2)))$.

Let us define a sequence $0 < \varepsilon_1 < \cdots < \varepsilon_n$ by setting $\varepsilon_i := \varepsilon_0 / C^{n-i}$ where ε_0 is chosen small enough for Lemmas 5.1, 5.2 and 5.3 to apply and C is as in Lemma 5.1. We assume without loss of generality that $\varepsilon_0 < 10^{-10}$ and $C > 1$. Observe that this way we have $\varepsilon_{k+1} \leq C\varepsilon_k$ for all k. We will place disks $D_1, \dots, D_n$ one by one, subject to the demands:

($\mathcal{D}$-1) $\|p(D_i) - P_j\| < \varepsilon_i$ and $|r(D_i) - \rho_j| < \varepsilon_i$ if $i \equiv j \bmod 4$;

($\mathcal{D}$-2) D_i, D_j are not tangent for all $i \neq j$;

($\mathcal{D}$-3) If K kisses $D_{i_1}, D_{i_2}, D_{i_3}$ and K' kisses $D_{j_1}, D_{j_2}, D_{j_3}$ with $\{i_1, i_2, i_3\} \neq \{j_1, j_2, j_3\}$ then $K \neq K'$.

For $k = 1, \dots, n$ let d_k denote the number of distinct labelled graphs $G(D_1, \dots, D_k)$ that can be represented by disks $D_1, \dots, D_k$ placed in such a way that the demands **($\mathcal{D}$-1)-($\mathcal{D}$-3)** are satisfied. We will show that:

$$d_{k+1} \geq \lfloor k/4 \rfloor^3 \cdot d_k \qquad \text{for all } 4 \leq k \leq n-1. \tag{7}$$

Clearly, by computations analogous to (5), this implies the lower bound in Theorem 1.2. Thus, to prove the lower bound in Theorem 1.2, it suffices to establish (7).

Let us suppose that disks $D_1, \dots, D_k$ have already been placed subject to the demands **($\mathcal{D}$-1)-($\mathcal{D}$-3)**. Let us first assume that $k \equiv 3 \bmod 4$.

Let $\mathcal{T}$ denote the set of all tripes $\tau = (i_0, i_1, i_2) \in [k]^3$ with $(i_0, i_1, i_2) \equiv (0, 1, 2) \bmod 4$. Note that

$$|\mathcal{T}| \geq \lfloor k/4 \rfloor^3. \tag{8}$$

By Lemma 5.1, for each triple $\tau = (i_0, i_1, i_2) \in \mathcal{T}$, there is a unique disk K_τ kissing $D_{i_0}, D_{i_1}, D_{i_2}$; and K_τ satisfies $\|p(K_\tau) - P_3\|, |r(K_\tau) - \rho_3| < \varepsilon_{k+1}$. By **($\mathcal{D}$-3)** all these kissing disks $K_\tau : \tau \in \mathcal{T}$ are distinct. For notational convenience, let $z_\tau \in \mathbb{R}^3$ denote the point

$$z_\tau := (p(K_\tau), r(K_\tau)),$$

and let $\mathcal{C}_\tau \subseteq \mathbb{R}^3$ be defined by:

$$\mathcal{C}_{(i_0,i_1,i_2)} := \mathcal{C}(D_{i_0}) \cap \mathcal{C}(D_{i_1}) \cap \mathcal{C}(D_{i_2}).$$

Let $O \subseteq \mathbb{R}^3$ denote the set

$$O := B(P_3, \varepsilon_{k+1}) \times (\rho_3 - \varepsilon_{k+1}, \rho_3 + \varepsilon_{k+1}).$$

Claim 5.4 *There exist nonempty open sets $O_\tau \subseteq \mathcal{C}_\tau \cap O$ such that for all pairs $\tau \neq \sigma \in \mathcal{T}$ of distinct triples either $O_\tau \cap \mathcal{C}_\sigma = \emptyset$ or $O_\sigma \cap \mathcal{C}_\tau = \emptyset$.*

Proof of Claim 5.4: Consider two triples $\tau \neq \sigma \in \mathcal{T}$. Let us assume that $r(K_\tau) \leq r(K_\sigma)$. By lemma 5.2 we have $z_\tau \in \mathrm{cl}(\mathcal{C}_\tau)$ and $z_\tau \notin \mathrm{cl}(\mathcal{C}_\sigma)$ because otherwise z_σ would not be the unique lowest point of $\mathrm{cl}(\mathcal{C}_\sigma)$. Hence there is a $\delta(\tau,\sigma)$ such that $B(z_\tau;\delta(\tau,\sigma)) \cap \mathcal{C}_\sigma = \emptyset$. If $r(K_\tau) > r(K_\sigma)$ we simply set $\delta(\tau,\sigma) = 1$. Let us now define, for each $\tau \in \mathcal{T}$:

$$\delta(\tau) := \min_{\substack{\sigma\in\mathcal{T},\\ \sigma\neq\tau}} \delta(\tau,\sigma),$$

and let us set

$$O_\tau := B(z_\tau, \delta(\tau)) \cap \mathcal{C}_\tau \cap O.$$

Then $O_\tau \neq \emptyset$ since z_τ is a limit point of $\mathcal{C}_\tau$, and $B(z_\tau, \delta(\tau)), O$ are both open neighbourhoods of z_τ. And, if τ, σ are distinct then $O_\tau \cap \mathcal{C}_\sigma = \emptyset$ if $r(K_\tau) \leq r(K_\sigma)$, and $O_\sigma \cap \mathcal{C}_\tau = \emptyset$ otherwise (by definition of $\delta(\tau,\sigma)$ and $\delta(\sigma,\tau)$). ■

Let us set

$$\mathcal{K} := \{K : K \text{ kisses all of } D_{j_1}, D_{j_2}, D_{j_3} \text{ for some } 1 \leq j_1 < j_2 < j_3 \leq k\},$$

(Note that here we are taking the kissing disks of *all* triples, not just those in $\mathcal{T}$. Also note that $\mathcal{K}$ is finite since a triple has at most two kissing disks by Theorem 2.2.) And let us also set

$$W := \bigcup_{K\in\mathcal{K}} \partial\mathcal{C}(K) \cup \bigcup_{i=1}^{k} \partial\mathcal{C}(D_i).$$

Now we fix arbitrary points

$$(q_\tau, r_\tau) \in O_\tau \setminus W.$$

(Note $O_\tau \setminus W$ is nonempty as O_τ is open and W has measure zero.) We have

Claim 5.5 *If we set $D_{k+1} := B(q_\tau, r_\tau)$ then* **($\mathcal{D}$-1)**-**($\mathcal{D}$-3)** *are satisfied by $D_1, \dots, D_{k+1}$.*

Proof of Claim 5.5: That **($\mathcal{D}$-1)** is satisfied follows immediately from the fact that $(q_\tau, r_\tau) \in O$. Similarly, **($\mathcal{D}$-2)** is satisfied since $(q_\tau, r_\tau) \notin \partial\mathcal{C}(D_i)$ for all $1 \leq i \leq k$ (by definition of W).

Now suppose that there are two distinct triples $t, s \in \binom{[k+1]}{3}$ such that $D_i : i \in t \cup s$ all kiss a common disk K. Since t, s are distinct we must have $|(t \cup s) \setminus \{k+1\}| \geq 3$. If $|(t \cup s) \setminus \{k+1\}| \geq 4$ then K kisses four of $D_1, \dots, D_k$, so that $D_1, \dots, D_k$ violates **($\mathcal{D}$-3)**. Hence we must have $|(t \cup s) \setminus \{k+1\}| = 3$. This implies that there are indices $1 \leq i_1, i_2, i_3 \leq k$ such that K kisses all of $D_{i_1}, D_{i_2}, D_{i_3}$ and D_{k+1}. In particular we have $K \in \mathcal{K}$ and hence $\partial\mathcal{C}(K) \subseteq W$. But we chose $(q_\tau, r_\tau) \notin W$ for all $\tau \in \mathcal{T}$. So D_{k+1} and K cannot kiss after all. This contradiction proves that **($\mathcal{D}$-3)** is also satisfied. ■

For $\tau \in \mathcal{T}$ let $N_\tau \subseteq [k]$ denote the set of neighbours of $k+1$ in the graph $G(D_1, \dots, D_k, B(q_\tau, r_\tau))$. We will prove:

Claim 5.6 *If $\tau \neq \sigma \in \mathcal{T}$ then $N_\tau \neq N_\sigma$.*

Proof of Claim 5.6: Observe that for any τ, σ we have $\sigma \subseteq N_\tau$ if and only if $(q_\tau, r_\tau) \in \mathcal{C}_\sigma$ (by definition of $\mathcal{C}_\sigma$). Now let $\tau \neq \sigma \in \mathcal{T}$ be arbitrary. By Claim 5.4 and because we chose $(q_\tau, r_\tau) \in O_\tau$ and $(q_\sigma, r_\sigma) \in O_\sigma$, we have $\tau \subseteq N_\tau, \sigma \subseteq N_\sigma$ and either $\tau \not\subseteq N_\sigma$ or $\sigma \not\subseteq N_\sigma$. ■

By (8) and Claim 5.6 we have

$$d_{k+1} \geq |\mathcal{T}| \cdot d_k \geq \lfloor k/4 \rfloor^3 d_k,$$

proving (7) in the case when $(k+1) \equiv 3 \bmod 4$. The cases when $(k+1) \equiv 0,1,2 \bmod 4$ can be handled by a straightforward modification of the $(k+1) \equiv 3 \bmod 4$ proof we just gave. First note, that by symmetry considerations we only need to consider the case when $(k+1) \equiv 0 \bmod 4$. Now we need to define $\mathcal{C}_{(a,b,c)} := \mathcal{C}(D_a) \setminus (\mathrm{cl}(\mathcal{C}(D_b)) \cup \mathrm{cl}(\mathcal{C}(D_c))$ for $(a,b,c) \equiv (3,1,2) \bmod 4$; and rather than Lemma 5.2 we use Lemma 5.3 in Claim 5.4. And, in the proof of Claim 5.6 we now prove that if $\tau = (a,b,c) \neq \sigma = (a',b',c')$ then $N_\tau \cap \tau = \{a\}, N_\sigma \cap \sigma = \{a'\}$ while either $N_\tau \cap \sigma \neq \{a'\}$ or $N_\sigma \cap \tau \neq \{a\}$.

It only remains to prove Lemmas 5.1, 5.2 and 5.3.

Proof of Lemma 5.1: Let us assume that $i = 3$ (the other cases are completely analogous). Let us write $p_j := p(D_j)$ and $r_j = r(D_j)$ for $j = 0,1,2$. Let us also define x_j, y_j by $(x_j, y_j) =: P_j - p_j$ and let us write $u = (x_0, x_1, x_2, y_0, y_1, y_2, r_0, r_1, r_2)$ for notational convenience. By Theorem 2.2, there is a disk of radius s kissing D_0, D_1, D_2 if and only if s solves a quadratic:

$$\alpha s^2 + \beta s + \gamma = 0, \tag{9}$$

where $\alpha = \alpha(u), \beta = \beta(u), \gamma = \gamma(u)$ can be expressed in terms of the parameters $x_0, \dots, r_2$ using the expressions given in Theorem 2.2. (In the language of Theorem 2.2 we have $\alpha = a, \beta = 2ar_1 + b, \gamma = ar_1^2 + br_1 + c$.) Note that α, β, γ are well-defined as long as the points $p_0 = P_0 + (x_0, y_0), p_1 = P_1 + (x_1, y_1), p_2 = P_2 + (x_2, y_2)$ are distinct (so that the distances d_{ij} occurring in the denominators are non-zero). By the quadratic formula, provided α, β, γ are well-defined and $\alpha \neq 0$, the two (complex) roots to equation (9) are

$$t = (-\beta + \sqrt{\beta^2 - 4\alpha\gamma})/2\alpha, \quad r = (-\beta - \sqrt{\beta^2 - 4\alpha\gamma})/2\alpha.$$

Let us set $u_0 := (0,0,0,0,0,0,1,1,1)$. It can be easily read off from the expressions for a, b, c from Theorem 2.2 that

$$\alpha(u_0) = -\frac{3}{4}, \quad \beta(u_0) = -\frac{3}{2}, \quad \gamma(u_0) = \frac{1}{4},$$

so that

$$t(u_0) = -\frac{2}{3}\sqrt{3} - 1, \quad r(u_0) = \frac{2}{3}\sqrt{3} - 1 = \rho_3.$$

Let us now pick a small $\varepsilon_0 > 0$ and set:

$$U := [-\varepsilon_0, \varepsilon_0]^6 \times [1 - \varepsilon_0, 1 + \varepsilon_0]^3.$$

Observe that, if ε_0 was chosen sufficiently small, then α, β, γ are well-defined for all $u \in U$. What is more, α, β, γ are in fact continuously-differentiable with respect to each of the parameters x_j, y_j, r_j on U.

By restricting to an even smaller value of ε_0 if necessary, we can assume without loss of generality that $\alpha(u) \neq 0$ and $\beta^2(u) - 4\alpha(u)\gamma(u) > 0$ for all $u \in U$ (recall that α, β, γ are continuous on U and $u_0 \in U$). Thus $t = t(u)$ and $r = r(u)$ are well-defined and real for all $u \in U$. Moreover, since $t(u), r(u)$ are continuous in u we can assume without loss of generality that $t(u) < 0$ and $r(u) > 0$ for all $u \in U$ (by restricting to an even smaller value of ε_0 if needed). Since $r(u)$ is the only non-negative solution of (9) and p_1, p_2, p_3 are clearly never collinear for $u \in U$ (having chosen ε_0 small enough), there is a unique kissing disk for all $u \in U$. Let us now observe that we can express the partial derivatives of r with respect to each of the parameters x_j, y_j, r_j in terms of the partial derivatives of α, β, γ, so that all partial derivatives $\frac{\partial r}{\partial x_j}, \frac{\partial r}{\partial y_j}, \frac{\partial r}{\partial r_j}$ are continuous on U (here we also use that $\alpha \neq 0, \beta^2 - 4\alpha\gamma > 0$ on U). Since U is compact, there exists a constant K such that all partial derivatives lie between $-K$ and K on U. This shows that, for all $0 < \varepsilon < \varepsilon_0$, if $\|p_j - P_j\| < \varepsilon$ and $|r_j - 1| < \varepsilon$ for $j = 0,1,2$ then

$$|r(u) - \rho_3| = |r(u) - r(u_0)| < 6K\varepsilon.$$

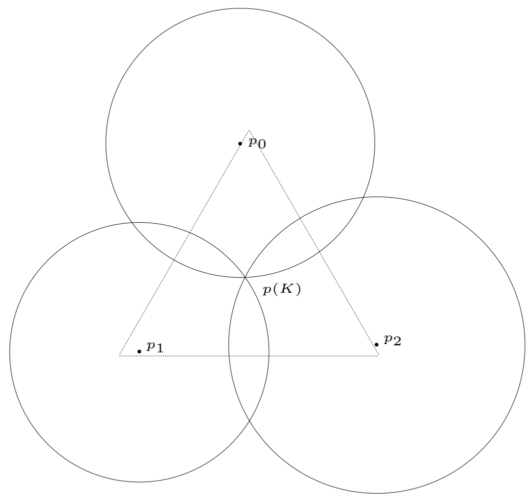

Figure 5: The circles $S(p_0, s_0), S(p_1, s_1), S(p_2, s_2)$ meet in the point $p(K)$.

As observed above there is a unique kissing disk for all $u \in U$ and its radius is $r(u)$. By Lemma 2.1 its center must be one of the following two points:

$$\begin{aligned} q &= p_1 + \left(\tfrac{d_{12}^2-(r_2+r)^2+(r_1+r)^2}{2d_{12}^2}\right)(p_2-p_1) \\ &\quad -\sqrt{\tfrac{4d_{12}^2(r_1+r)^2-(d_{12}^2-(r_2+r)^2+(r_1+r)^2)^2}{4d_{12}^4}}\begin{pmatrix} (p_1)_y-(p_2)_y \\ (p_2)_x-(p_1)_x \end{pmatrix}, \\ p &= p_1 + \left(\tfrac{d_{12}^2-(r_2+r)^2+(r_1+r)^2}{2d_{12}^2}\right)(p_2-p_1) \\ &\quad +\sqrt{\tfrac{4d_{12}^2(r_1+r)^2-(d_{12}^2-(r_2+r)^2+(r_1+r)^2)^2}{4d_{12}^4}}\begin{pmatrix} (p_1)_y-(p_2)_y \\ (p_2)_x-(p_1)_x \end{pmatrix}. \end{aligned}$$

(Here we set $d_{12} = \|p_1 - p_2\|$.) One can easily check that $q(u_0) = -P_3, p(u_0) = P_3$. Similarly to before, all partial derivatives of the coordinates of $q(u), p(u)$ with respect to the parameters x_j, y_j, r_j are continuous on U, and hence bounded in absolute value by some constant L, so that

$$\begin{aligned} \|p - P_3\| &= \|p(u) - p(u_0)\| < 12L\varepsilon, \\ \|q + P_3\| &= \|q(u) - q(u_0)\| < 12L\varepsilon. \end{aligned}$$

We see that $\|q(u)-P_0\| > \sqrt{3} > r_0+r(u)$ for all $u \in U$ (provided we chose ε_0 sufficiently small), so that the unique kissing disk is $K = B(p(u), r(u))$ for all $u \in U$. Hence if we set $C := \max(6K, 12L)$ then we have proved the lemma. ■

Proof of Lemma 5.2: For convenience, let us write $s_j = r(K) + r(D_j)$ for $j = 0, 1, 2$. Observe that $p(K)$ is the unique intersection point of $S(p_0, s_0)$, $S(p_1, s_1)$, $S(p_2, s_2)$. (See figure 5.) Moreover, provided ε_0 was chosen sufficiently small, we have $B(p_0, s_0) \cap B(p_1, s_1) \cap B(p_2, s_2) = \emptyset$ and $\mathrm{cl}(B(p_0, s_0)) \cap \mathrm{cl}(B(p_1, s_1)) \cap \mathrm{cl}(B(p_2, s_2)) = \{p(K)\}$. (This follows for instance from the fact that – ε_0 being small enough – $p(K)$ lies in the interior of the triangle $\mathrm{conv}(\{p_0, p_1, p_2\})$.)

Now pick an arbitrary $(p, r) \in \mathrm{cl}(\mathcal{C}(D_0) \cap \mathcal{C}(D_1) \cap \mathcal{C}(D_2))$. First suppose that $p \neq p(K)$. Then $p \notin \mathrm{cl}(B(p_i, s_i))$ for at least one of $i = 0, 1, 2$. In other words $\|p-p_i\| > s_i$. Since $(p, r) \in \mathrm{cl}(\mathcal{C}(D_i))$ we must have $s_i < \|p - p_i\| \leq r + r_i$. This gives that $r(K) < r$, by definition of s_i.

Similarly, if $p = p(K)$ then we must have $r \geq r(K)$. ■

Proof of Lemma 5.3: Let us again write $s_j = r(K) + r(D_j)$ (for $j = 1, 2, 3$ this time). Again $p(K)$ is the unique intersection point of $S(p_1, s_1), S(p_2, s_2), S(p_3, s_3)$. (See figure 6.)

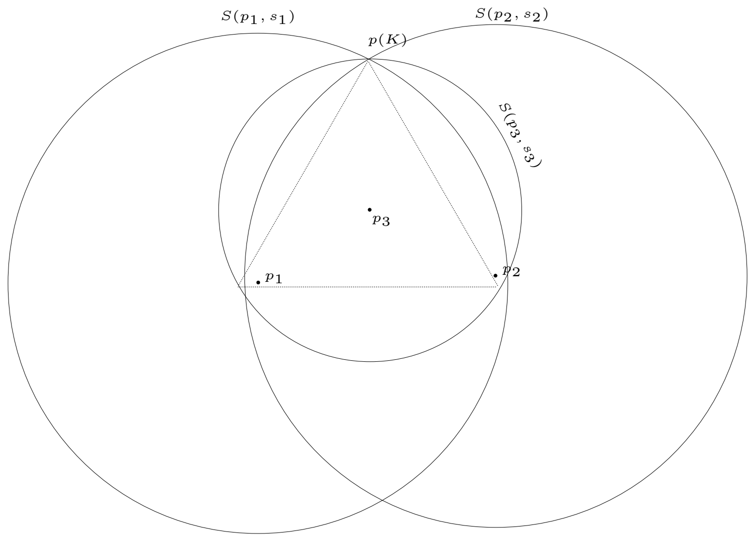

Figure 6: The circles $S(p_1, s_1), S(p_2, s_2), S(p_3, s_3)$ meet in the point $p(K)$.

Observe that, provided ε_0 was chosen sufficiently small, $B(p_3, s_3) \subseteq B(p_1, s_1) \cup B(p_2, s_2)$. (This can be seen from the facts that – provided ε_0 was sufficiently small – we have $s_3 < s_1, s_2$ and $p_3 \in \operatorname{conv}(\{p_1, p_2, p(K)\})$.)

Now pick an arbitrary $(p, r) \in \operatorname{cl}(\mathcal{C}(D_3) \setminus (\mathcal{C}(D_1) \cup \mathcal{C}(D_2)))$.

Let us first suppose that $p = p(K)$. Since $(p, r) \notin \mathcal{C}(D_1)$ we have $s_1 = \|p - p_1\| \geq r + r_1$. It follows that $r \leq r(K)$.

Next, let us suppose that $p \in B(p_1, s_1)$. Since $(p, r) \notin \mathcal{C}(D_1)$, we have $\|p - p_1\| \geq r + r_1$. Hence we have $r \leq \|p - p_1\| - r_1 < s_1 - r_1 = r(K)$.

Similarly, if $p \in B(p_2, s_2)$ then $r < r(K)$.

Let us thus suppose that $p \notin B(p_1, s_1) \cup B(p_2, s_2) \cup \{p(K)\}$. Since $B(p_3, s_3) \subseteq B(p_1, s_1) \cup B(p_2, s_2)$ we immediately get that

$$\operatorname{dist}(p, B(p_3, s_3)) \geq \operatorname{dist}(p, B(p_1, s_1) \cup B(p_2, s_2)).$$

Moreover, by considering the angles at which $S(p_1, s_1), S(p_2, s_2), S(p_3, s_3)$ intersect, we see that in fact $\operatorname{dist}(p, B(p_3, s_3)) > \operatorname{dist}(p, B(p_1, s_1) \cup B(p_2, s_2))$. We can assume without loss of generality $\operatorname{dist}(p, B(p_3, s_3)) > \operatorname{dist}(p, B(p_1, s_1))$.

Let us also observe that we have $\operatorname{dist}(p, B(p_i, s_i)) + s_i = \|p - p_i\|$ for $i = 1, 2, 3$. Since $(p, r) \in \operatorname{cl}(\mathcal{C}(D_3))$ we must have $\|p - p_3\| \leq r + r_3$. Hence $r \geq \|p - p_3\| - r_3 = \operatorname{dist}(p, B(p_3, s_3)) + s_3 - r_3 = \operatorname{dist}(p, B(p_3, s_3)) + r(K)$.

On the other hand, since $(p, r) \notin \mathcal{C}(D_1)$ we must have $\|p - p_1\| \geq r + r_1$. In other words, $r \leq \|p - p_1\| - r_1 = \operatorname{dist}(p, B(p_1, s_1)) + s_1 - r_1 = \operatorname{dist}(p, B(p_1, s_1)) + r(K)$. But this gives $\operatorname{dist}(p, B(p_3, s_3)) + r(K) \leq r \leq \operatorname{dist}(p, B(p_1, s_1)) + r(K)$, contradicting our assumption that $\operatorname{dist}(p, B(p_3, s_3)) > \operatorname{dist}(p, B(p_1, s_1))$.

We see that there is no $(p, r) \in \operatorname{cl}(\mathcal{C}(D_3) \setminus (\mathcal{C}(D_2) \cup \mathcal{C}(D_3)))$ with $p \notin B(p_1, s_1) \cup B(p_1, s_2) \cup \{p(K)\}$. ■

6 Unit disk graphs with bounded clique size

For a unit disk graph G, the maximum degree Δ and the maximum clique size ω are related by $\Delta/6 \leq \omega - 1 \leq \Delta$. These inequalities are well known and not hard to see. Are there many unit disk graphs with small maximum clique size?

To set the scene, let us first observe that there are at least exponentially many triangle-free unlabelled unit disk graphs. For, we may line up about $2n/3$ barely touching unit disks to form a path, pick any $n/3$ of these disks and to each one add a new unit disk touching just it. This yields $2^{(\frac{2}{3}+o(1))n}$ distinct triangle-free unlabelled unit disk graphs.

Theorem 6.1 *Let a non-decreasing bound $b = b(n)$ be given, and let $\mathcal{U}_n$ denote the set of unlabelled unit disk graphs on n vertices with maximum clique size at most b. Then*

$$|\mathcal{U}_n| \leq 2^{12\,(b+1)n}.$$

In particular, if b is a constant then $\mathcal{U}_n$ has just exponential growth. The number of labelled unit disk graphs on n vertices with maximum clique size at most b is at most $n!\,|\mathcal{U}_n|$. These upper bounds together with the earlier lower bounds show the scarcely surprising result that for unit disk graphs, in both the unlabelled and the labelled cases, there is a $\delta > 0$ such that the proportion of graphs with maximum clique size at least $\delta \ln n$ tends to 1 as $n \to \infty$. (We may take $\delta = \frac{1}{13}$.)

To prove this theorem we use one basic lemma, see for example the proof of Lemma 2.1 in [3].

Lemma 6.2 *In a graph with maximum degree at most d, there are at most $(ed)^n$ connected induced subgraphs with n vertices containing any given vertex.*

Proof of Theorem 6.1: Let $\mathcal{C}_n$ denote the set of connected graphs in $\mathcal{U}_n$. Let us first observe the (loose) bound that

$$|\mathcal{U}_n| < 2^n |\mathcal{C}_n|. \tag{10}$$

For, given a graph $G \in \mathcal{U}_n$ with $k \geq 2$ components, we may pick a representation where the components are lined up say along the x-axis, and then slide the components together until we just form a connected graph G', with exactly $k-1$ new bridges between the old components. (We may ensure that nothing more complicated happens by perturbing the points if necessary.) From G' with the new bridges deleted we obtain the original graph G, so G' is obtained in this way at most 2^t times if it has t bridges. Finally note that every graph with n vertices has $< n$ bridges, which completes the proof of (10).

Let s satisfy $\frac{1}{2} < s < 2^{-\frac{1}{2}}$. Partition $\mathbb{R}^2$ into squares $C_{\mathbf{z}}$ of side s in a natural way, by setting $C_{\mathbf{z}} = [sz_1, s(z_1+1)) \times [sz_2 + s(z_2+1))$ for each $\mathbf{z} = (z_1, z_2) \in \mathbb{Z}^2$. Form the graph H with vertex set the set of all these squares, by joining two distinct squares if they contain points at distance at most 1. Then H is regular of degree 24: for example the square $S_{(2,2)}$ is adjacent to each other square $S_{(i,j)}$ with $0 \leq i, j \leq 4$. Fix $S_0 = S_{(0,0)} = [0, s)^2$ as a reference square. Let $\mathcal{H}_n$ denote the set of connected induced subgraphs of H of order n which contain S_0. By Lemma 6.2

$$|\mathcal{H}_n| \leq (e \cdot 24)^n < (2^3 \cdot 9)^n. \tag{11}$$

Consider a graph G in $\mathcal{C}_n$. Let the points $p_1, \ldots, p_n$ in the plane give a representation of G, where p_1 is in S_0. Then the squares $C_{\mathbf{z}}$ which contain at least one of the points p_i induce a connected subgraph of H, which can be extended if necessary to a graph G' in $\mathcal{H}_n$. For each graph G in $\mathcal{C}_n$ let us fix one such graph G' in $\mathcal{H}_n$.

Let H_0 be any graph in $\mathcal{H}_n$. How many graphs $G \in \mathcal{C}_n$ can have $G' = H_0$? Let us rename the vertices (squares) of H_0 as the *boxes* $B_1, \ldots, B_n$ where B_1 is S_0. For distinct $i, j \in [n]$ let $a_{ij} = 1$ if the boxes B_i and B_j are adjacent in H_0, and let $a_{ij} = 0$ otherwise.

To form a graph G in $\mathcal{C}_n$ which might have $G' = H_0$, we first divide n unlabelled points amongst these n boxes. The number of ways to do this (with no further restriction) is $\binom{2n-1}{n-1} \leq 4^n$. Consider such an allocation, where n_i is the number of points in B_i for each i. Observe that each $n_i \leq b$, since any two points in the same box are at distance < 1 apart. At this stage we have decided which box each point is in but not where it is placed in the box. Thus the number of 'undecided' edges between distinct boxes B_i and B_j is $a_{ij} n_i n_j$, and so the total number of 'undecided' edges is

$$\frac{1}{2} \sum_{i,j} a_{ij} n_i n_j \leq \frac{1}{2} \sum_{i,j} a_{ij} n_i \cdot b \leq 12\, bn.$$

It follows that, given the choice of $n_1, \ldots, n_n$, the number of graphs in $\mathcal{C}_n$ we can form is at most $2^{12\,bn}$. As we noted before, there are at most 4^n choices for $n_1, \ldots, n_n$. Hence the number of graphs $G \in \mathcal{C}_n$ which could have $G' = H_0$ is at most $4^n 2^{12\,bn}$. Therefore by (11)

$$|\mathcal{C}_n| \leq 4^n 2^{12\,bn} |\mathcal{H}_n| \leq (2^5 \cdot 9)^n \, 2^{12\,bn}.$$

Hence by (10)

$$|\mathcal{U}_n| \leq (2^6 \cdot 9)^n \, 2^{12\,bn} \leq 2^{12\,(b+1)n}$$

and we are done. ■

A similar bound holds, with a similar proof, for the intersection graphs of any convex body in $\mathbb{R}^d$.

7 Concluding remarks

The case of $d = 2$ dimensions seems to be the most natural, but we could consider other dimensions. We believe that constructions similar to the ones we gave above for $d = 2$ will show that the number of intersection graphs of d-dimensional unit balls is $n^{dn}\Theta(1)^n$, and the number of intersection graphs of d-dimensional balls is $n^{(d+1)n}\Theta(1)^n$, where the upper bounds again follow easily from Warren's theorem. Further, it would be natural to ask if more precise results like those proposed in Conjectures 1.3 and 1.4 hold here.

Now let us fix $d = 1$, and consider the class $\mathcal{UIG}$ of unit interval graphs. Let $\mathcal{UIG}(n)$ denote the number of such graphs on vertex set $[n]$. The following result is in line with Conjecture 1.3.

Theorem 7.1 *There is a finite constant $\alpha > 0$ such that*

$$\mathcal{UIG}(n) = (\alpha + o(1))^n n^n.$$

Proof: We use a supermultiplicativity argument. Observe that, by Stirling's approximation, the result we want to prove is equivalent to there being a finite constant $\gamma > 0$ such that $\mathcal{UIG}(n) = (\gamma + o(1))^n n!$ (with $\gamma = \alpha e$); that is, $\mathcal{UIG}$ has (exponential) growth constant γ. Now by the 'exponential formula', see for example the text book by Aigner [1], it suffices to show that the class $\mathcal{C}$ of connected unit interval graphs has growth constant γ; for in this case, the exponential generating functions for $\mathcal{C}$ and for $\mathcal{UIG}$ have the same radius of convergence γ^{-1}, and so $\mathcal{UIG}(n) \leq (\gamma + o(1))^n n!$.

Fix positive integers i and j. We may form a connected unit interval graph G on vertex set $V = \{1, .., i+j\}$ by choosing a set A of i vertices; choosing a connected unit interval graph G_1 on A and a connected unit interval graph G_2 on $V \setminus A$; putting a representation of G_1 on the line and a representation of G_2 on the line to the right of G_1; and then moving them together until the nearest vertices, say v_1 in G_1 and v_2 in G_2 are at distance 1 (or 1-). Observe that $v_1 v_2$ now forms a bridge. (We may assume that vertices are represented by distinct points.)

Let c_n denote the number of connected unit interval graphs on $[n]$. Then there are $\binom{i+j}{i} c_i c_j$ constructions. Further, each unit interval graph is constructed at most twice, by the following claim.

Claim 7.2 *Let G be a connected unit interval graph and let i be a positive integer. Then G has at most two oriented bridges uv such that the component of $G \setminus uv$ containing u has exactly i vertices, and in any representation of G any such component consists of the leftmost i vertices or the rightmost i vertices.*

Proof of Claim 7.2: To prove the claim, observe that if uv is a bridge, and G_u and G_v are the components of $G \setminus uv$ containing u and v respectively, then in every representation of G such that the point representing u is to the left of the point representing v, all of G_u is placed to the left of all of G_v (since no vertex in G_u other than u is adjacent to v). Thus if G_u has i vertices, it consists of the leftmost i vertices in the representation. ■

Now that we know that the claim holds, we have $c_{i+j} \geq \binom{i+j}{i} c_i c_j / 2$. Let $f(n) = c_n/(2 \cdot n!)$. Then $f(i+j) \geq f(i)f(j)$ for all positive integers i and j; that is, f is supermultiplicative. Further we know from Warren's Theorem that $c_n \leq a^n n^n$ for some constant a, and so $f(n) < (ae)^n$. Now we can use a standard result, Fekete's Lemma, on supermultiplicative functions. It follows that $f(n)^{1/n} \to \gamma$ as $n \to \infty$, for some $0 < \gamma < \infty$. Thus, as $n \to \infty$, $(c_n/n!)^{1/n} \to \gamma$ and so $c_n = (\alpha + o(1))^n n^n$ where $\alpha = \gamma/e$; and finally $\mathcal{UIG}(n) = (\alpha + o(1))^n n^n$, as we noted earlier. ■

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Appendix : The problem of the disk kissing three given disks

Proof of Theorem 2.2: Let P_i denote the center of D_i for $i = 1, 2, 3$. We will start with the proof of part **(i)**, which we will do "inside-out". Instead of starting with the centers P_1, P_2, P_3 of the three disks fixed in the plane and trying to place the fourth circle, we start by fixing the center of the fourth disk on the origin O and vary its radius $s > 0$. We wish to find a value for s for which we can place points Q_1, Q_2, Q_3 in the plane with $\mathrm{dist}(O, Q_i) = s + r_i$ and $\mathrm{dist}(Q_i, Q_j) = d_{ij}$. This is equivalent to the original problem of finding a disk kissing D_1, D_2, D_3, because once we have found s and Q_1, Q_2, Q_3, we can find an isometry that maps Q_i to P_i for all i. This isometry will transform the circle of radius s around the origin O into the sought circle kissing D_1, D_2, D_3.

Without loss of generality, we can choose Q_1 on the negative x-axis, so that it has coordinates $(-(s + r_1), 0)$. See figure 7 for an illustration to go with the proof.
Let (x_2, y_2) be the coordinates of Q_2. Since the distance between Q_1 and Q_2 must equal d_{12}, it holds that:

$$(s + r_1 + x_2)^2 + y_2^2 = d_{12}^2. \tag{12}$$

.
And, because the distance between O and Q_2 must equal $s + r_2$:

$$x_2^2 + y_2^2 = (s + r_2)^2. \tag{13}$$

By combining (12) and (13) we get:

$$\begin{aligned} d_{12}^2 &= (s + r_1)^2 + 2x_2(s + r_1) + x_2^2 + y_2^2 \\ &= (s + r_1)^2 + 2x_2(s + r_1) + (s + r_2)^2. \end{aligned}$$

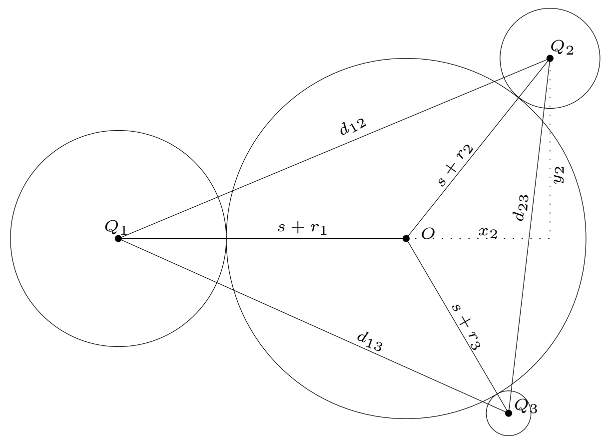

Figure 7: The picture to go with the proof of part **(i)** of Theorem 2.2.

Rearranging, we see that

$$x_2 = \frac{d_{12}^2 - (s+r_2)^2 - (s+r_1)^2}{2(s+r_1)} \tag{14}$$

Let us set $\alpha_2 := \angle OQ_1Q_2$. Then

$$\begin{aligned}
\cos(\alpha_2) &= \frac{x_2+s+r_1}{d_{12}} = \frac{d_{12}^2-(s+r_2)^2-(s+r_1)^2+2(s+r_1)^2}{2(s+r_1)d_{12}} \\
&= \frac{d_{12}^2+(s+r_1)^2-(s+r_2)^2}{2(s+r_1)d_{12}} = \frac{d_{12}^2+(r_1-r_2)(2s+r_1+r_2)}{2(s+r_1)d_{12}} \\
&= \frac{d_{12}^2+(r_1-r_2)(2s+2r_1+r_2-r_1)}{2(s+r_1)d_{12}} \\
&= \frac{d_{12}^2+(r_1-r_2)2(s+r_1)-(r_1-r_2)^2}{2(s+r_1)d_{12}} \\
&= \frac{r_1-r_2}{d_{12}} + \frac{d_{12}^2-(r_1-r_2)^2}{2d_{12}(s+r_1)}.
\end{aligned} \tag{15}$$

Completely analogously, setting $\alpha_3 := \angle OQ_1Q_3$ we have

$$\cos(\alpha_3) = \frac{r_1-r_3}{d_{13}} + \frac{d_{13}^2-(r_1-r_3)^2}{2d_{13}(s+r_1)}. \tag{16}$$

Let us denote $\alpha := \angle P_2P_1P_3$. We would like to choose Q_1, Q_2, Q_3 such that $\angle Q_2Q_1Q_3 = \alpha$. Note that since $\mathrm{dist}(Q_1,Q_2) = d_{12}$ and $\mathrm{dist}(Q_1,Q_3) = d_{13}$ this is equivalent to $\mathrm{dist}(Q_2,Q_3) = d_{23}$. We certainly need to have:

$$\alpha_2 \pm \alpha_3 = \pm\alpha \pmod{2\pi}. \tag{17}$$

Or, rewritten in a form that is more convenient for our computation, we have

$$\alpha_2 \pm \alpha = \pm\alpha_3 \pmod{2\pi}. \tag{18}$$

Taking cosines on both sides of this last equation, we get

$$\cos(\alpha_2)\cos(\alpha) \mp \sin(\alpha_2)\sin(\alpha) = \cos(\alpha_3). \tag{19}$$

(Here we have used the identities $\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y)$ and $\cos(-x) = \cos(x)$.) Rearranging, we get

$$\cos(\alpha)\cos(\alpha_2) - \cos(\alpha_3) = \pm\sin(\alpha)\sin(\alpha_2). \tag{20}$$

Taking squares on both sides,

$$\cos^2(\alpha)\cos^2(\alpha_2) - 2\cos(\alpha)\cos(\alpha_2)\cos(\alpha_3) + \cos^2(\alpha_3) = \sin^2(\alpha)\sin^2(\alpha_2).$$

Using that $\sin^2(\alpha)\sin^2(\alpha_2) = \sin^2(\alpha) - \sin^2(\alpha)\cos^2(\alpha_2)$ and $\cos^2(\alpha)+\sin^2(\alpha) = 1$, we can rewrite this last equation to:

$$\cos^2(\alpha_2) + \cos^2(\alpha_3) - 2\cos(\alpha)\cos(\alpha_2)\cos(\alpha_3) - \sin^2(\alpha) = 0 \tag{21}$$

Substituting (15) and (16) into (21):

$$\begin{aligned} 0 \quad = \quad & \left(\tfrac{r_1-r_2}{d_{12}} + \tfrac{d_{12}^2-(r_1-r_2)^2}{2d_{12}(s+r_1)}\right)^2 + \left(\tfrac{r_1-r_3}{d_{13}} + \tfrac{d_{13}^2-(r_1-r_3)^2}{2d_{13}(s+r_1)}\right)^2 \\ & -2\cos(\alpha)\left(\tfrac{r_1-r_2}{d_{12}} + \tfrac{d_{12}^2-(r_1-r_2)^2}{2d_{12}(s+r_1)}\right)\left(\tfrac{r_1-r_3}{d_{13}} + \tfrac{d_{13}^2-(r_1-r_3)^2}{2d_{13}(s+r_1)}\right) - \sin^2(\alpha). \end{aligned}$$

If we multiply the previous equation with $(s+r_1)^2$ we indeed get a quadratic equation:

$$a(s+r_1)^2 + b(s+r_1) + c = 0, \tag{22}$$

where the coefficients a, b, c are given by:

$$\begin{aligned} a = \quad & \left(\tfrac{r_1-r_2}{d_{12}}\right)^2 + \left(\tfrac{r_1-r_3}{d_{13}}\right)^2 - 2\cos(\alpha)\left(\tfrac{r_1-r_2}{d_{12}}\right)\left(\tfrac{r_1-r_3}{d_{13}}\right) - \sin^2(\alpha), \\ b = \quad & 2\left(\tfrac{r_1-r_2}{d_{12}}\right)\left(\tfrac{d_{12}^2-(r_1-r_2)^2}{2d_{12}}\right) + 2\left(\tfrac{r_1-r_3}{d_{13}}\right)\left(\tfrac{d_{13}^2-(r_1-r_3)^2}{2d_{13}}\right) \\ & -2\cos(\alpha)\left[\left(\tfrac{r_1-r_2}{d_{12}}\right)\left(\tfrac{d_{13}^2-(r_1-r_3)^2}{2d_{13}}\right) + \left(\tfrac{r_1-r_3}{d_{13}}\right)\left(\tfrac{d_{12}^2-(r_1-r_2)^2}{2d_{12}}\right)\right], \\ c = \quad & \left(\tfrac{d_{12}^2-(r_1-r_2)^2}{2d_{12}}\right)^2 + \left(\tfrac{d_{13}^2-(r_1-r_3)^2}{2d_{13}}\right)^2 \\ & -2\cos(\alpha)\left(\tfrac{d_{12}^2-(r_1-r_2)^2}{2d_{12}}\right)\left(\tfrac{d_{13}^2-(r_1-r_3)^2}{2d_{13}}\right). \end{aligned} \tag{23}$$

By the cosine rule $\cos(\alpha) = (d_{12}^2 + d_{13}^2 - d_{23}^2)/2d_{12}d_{13}$. This enables us to rewrite a, b, c into the expressions given in the theorem. This proves part **(i)**.

We now turn attention to parts **(ii)** and **(iii)**. We will first show that, for any $s > 0$ that solves (2), a disk of radius s can be constructed that kisses all three of D_1, D_2 and D_3. Let x_2 be given by (14) and let x_3 be defined in the corresponding way. We will choose either $y_2 := -\sqrt{(s+r_2)^2 - x_2^2}$ or $y_2 := +\sqrt{(s+r_2)^2 - x_2^2}$ and we choose y_3 similarly. This way it will certainly hold that $\alpha_2 = \angle OQ_1Q_2$ satisfies (15) and $\alpha_3 = \angle OQ_1Q_3$ satisfies (16).

Let us observe that, to prove the existence of a kissing disk, it suffices to show that signs can be chosen for y_2, y_3 in such a way that $\angle Q_2Q_1Q_3 = \alpha$. We shall now work backwards through the computations to achieve this. Note that (21) must hold, since $t = s + r_1$ solves (22) and (22) is just a reformulation of (21). This last equation (21) is obtained by squaring (20) and rearranging terms. Since (20) does not determine the sign of its right-hand side, it must hold with our choice of x_1, y_1, x_2, y_2. Equation (19) must then also hold, as it is just a rearrangement of (20). Because (19) is obtained from (18) by taking cosines on both sides, and $\cos(\alpha) = \cos(\beta)$ if and only if $\alpha = \pm\beta (\bmod 2\pi)$, and the sign of the right-hand side of (18) is not determined and $\cos(-x) = \cos(x)$, we find that (18) holds. Hence (17) also holds as it is just a rearrangement of the terms of (18). We now pick signs for y_2, y_3 as follows. If $\alpha_2 - \alpha_3 = \alpha$ or $\alpha_2 - \alpha_3 = 2\pi - \alpha$ then we let y_2 and y_2 both be positive. If $\alpha_2 + \alpha_3 = \alpha$ or $\alpha_2 + \alpha_3 = 2\pi - \alpha$ then we let y_2 be positive and y_3 be negative. In this way we indeed get that $\angle Q_2Q_1Q_3 = \alpha$. (note that $\angle Q_2Q_1Q_3$

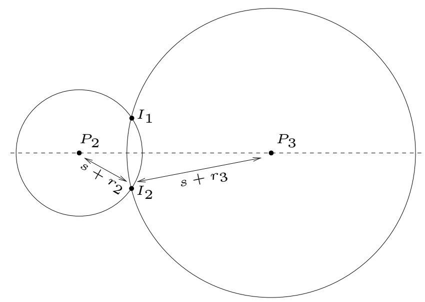

Figure 8: The picture to go with the proof of parts **(ii)** and **(iii)** of Theorem 2.2

may be facing away from the origin O in case α_2 and α_3 add up to the complement of α). This proves at least one kissing disk exists for each $s > 0$ that solves (2).

Now let $s > 0$ be a solution of (2). We know that a kissing disk with radius s exists. The center of a disk kissing D_1, D_2, D_3 must be an intersection point of $S(P_1, s+r_1), S(P_2, s+r_2)$ and $S(P_3, s+r_3)$. Because $S(P_2, s+r_2)$ and $S(P_3, s+r_3)$ intersect in at most two points I_1 and I_2(see figure 8), there are at most two kissing disks of radius s.

Let us now suppose that P_1, P_2, P_3 are collinear. Let ℓ denote the through P_1, P_2, P_3. If D is a kissing disk, then the disk D' we get by reflecting D through ℓ is also a kissing disk. (Note the center of D cannot lie on ℓ, so that D' is distinct from D.) Hence, if P_1, P_2, P_3 are collinear then there are exacty two kissing disks of radius s. To prove part **(ii)** we now need to show there cannot be any other solution of (2). Since P_1, P_2, P_3 are collinear, we have $\alpha = 0$ or $\alpha = \pi$. Let us write $\delta_i := \frac{d_{1i}^2-(r_1-r_i)^2}{2d_{1i}}$ and $\rho_i := \frac{r_1-r_i}{d_{1i}}$ for $i = 1, 2$ for notational convenience. If $\alpha = \pi$ then $\cos(\alpha) = -1, \sin(\alpha) = 0$, so that (23) gives

$$a = (\rho_2 + \rho_3)^2, \quad b = 2(\rho_2 + \rho_3)(\delta_2 + \delta_3), \quad c = (\delta_2 + \delta_3)^2.$$

Thus, we can rewrite (22) to

$$((\rho_2 + \rho_3)(s + r_1) + (\delta_2 + \delta_3))^2 = 0.$$

If $\alpha = 0$ the we get, similarly

$$((\rho_2 - \rho_3)(s + r_1) + (\delta_2 - \delta_3))^2 = 0.$$

Thus, if P_1, P_2, P_3 are collinear, then (2) has only one solution (of multiplicity two). This proves part **(ii)** of the theorem.

Now suppose that P_1, P_2, P_3 are not collinear, and let $s > 0$ be a solution of (2). Suppose there are two distinct kissing disks of radius s. Then their centers must be I_1, I_2 defined above and both I_1, I_2 must lie on $S(P_1, s + r_1)$. But then P_1 has equal distance to I_1, I_2 and it must therefore lie on the line through P_2 and P_3, a contradiction. Hence there is precisely one kissing disk with radius s, which proves part **(iii)** of the theorem. ∎