Dot product representations of planar graphs

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Abstract

A graph G on n vertices is a k-dot product graph if there are vectors $u_1, \ldots, u_n \in \mathbb{R}^k$, one for each vertex of G, such that $u_i^T u_j \geq t$ if and only if $ij \in E(G)$. Fiduccia, Scheinerman, Trenk and Zito asked whether every planar graph is a 3-dot product graph. We show that the answer is "no". On the other hand, every planar graph is a 4-dot product graph.

1 Introduction and statement of results

We study a type of geometric representation of graphs using vectors from $\mathbb{R}^k$ for some $k \in \mathbb{N}$. Let G be a graph with n vertices. We say G is a *k-dot product graph* if there exist vectors $u_1, \ldots, u_n \in \mathbb{R}^k$ such that $u_i^T u_j \geq 1$ if and only if $ij \in E(G)$. An explicit set of vectors in $\mathbb{R}^k$ that exhibits G in this way is called a *k-dot product representation* of G. The *dot product dimension* of G is the least k such that there is a k-dot product representation of G. (Every graph has finite dot product dimension, see for instance [4].)

The well-studied class of threshold graphs is closely related to 1-dot product graphs: a 1-dot product graph has at most two nontrivial connected components and each of these components is a threshold graph (see [4]). An extensive survey of threshold graphs is [7].

Notions closely related to dot product representations were studied in the context of communications complexity by amongst others Paturi and

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Simon [8], Alon, Frankl and Rödl [1] and Lovász [6]. Partially motivated by these works on communication complexity, the authors Reiterman, Rödl and Šiňajová [9, 10, 11] introduced dot product representations of graphs and studied them extensively. They obtained several bounds for the dot product dimension in terms of threshold dimension, sphericity, chromatic number, maximum degree, maximum average degree, and maximum complementary degree; they also detailed various examples. Here it should be mentioned that they used a slightly different definition: they used a threshold $t \in \mathbb{R}$ (i.e. $ij \in E(G)$ if and only if $u_i^T u_j \geq t$). This leads to a slightly larger class of graphs. However, in most of their constructions Reiterman et al. in fact take $t = 1$ and the proofs usually transfer easily to the special case $t = 1$. The proofs of our results below can also easily be adapted to work also for the more general definition of a dot product graph used in [9, 10, 11]. Fiduccia, Scheinerman, Trenk and Zito [4] considered amongst other things the dot product dimension of bipartite, complete multipartite and interval graphs.

Both Reiterman et al. [10] and Fiduccia et al. [4] proved that every forest is a 3-dot product graph. Envisioning a potential extension to this result, Fiduccia et al. asked whether every planar graph is a 3-dot product graph. Here we will answer this question in the negative by describing a counterexample. In contrast, we show that any planar graph has dimension at most 4.

Theorem 1 *Every planar graph is a* 4*-dot product graph, and there exist planar graphs which are not* 3*-dot product graphs.*

In the next section we develop some notation and recollect some spherical geometry needed in Section 3. In Section 3 we present a planar graph that is not 3-dot product graphs and in Section 4 we show that every planar graph has a 4-dot product representation.

2 Preliminaries

We shall review some basic geometry on the unit sphere S^2. For $u, v \in S^2$, let us denote by $[u, v]$ the (shortest) spherical arc between u and v. Let $\mathrm{dist}_{S^2}(u, v)$ denote the length of $[u, v]$. Then $\mathrm{dist}_{S^2}(u, v)$ equals the angle between the two vectors $u, v \in S^2$. It can thus be expressed as

$$\mathrm{dist}_{S^2}(u, v) = \arccos(v^T u).$$

For $r \geq 0$, let the *spherical cap* of radius r around $v \in S^2$ be defined as

$$\mathrm{cap}(v, r) := \{u \in S^2 : \mathrm{dist}_{S^2}(u, v) \leq r\}.$$

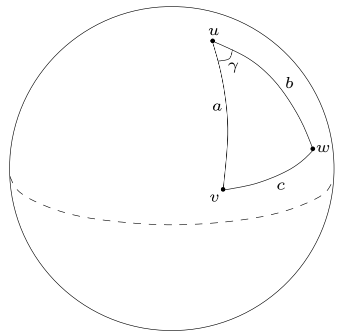


Figure 1: A spherical triangle.

Suppose that $u, v, w \in S^2$ are three points on the sphere in general position. We shall call the union of the three spherical arcs $[u, v], [v, w], [u, w]$ a *spherical triangle*. Let us write $a := \mathrm{dist}_{S^2}(u, v), b := \mathrm{dist}_{S^2}(u, w), c := \mathrm{dist}_{S^2}(v, w)$, and let γ denote the angle between $[u, v]$ and $[u, w]$. See Figure 1. Recall the spherical law of cosines:

$$\cos(c) = \cos(a)\cos(b) + \sin(a)\sin(b)\cos(\gamma). \tag{1}$$

The spherical law of cosines can be rephrased as

$$v^T w = (u^T v) \cdot (u^T w) + \cos(\gamma)\sqrt{(1 - (u^T v)^2)(1 - (u^T w)^2)}. \tag{2}$$

This second form will be more useful for our purposes.

Similarly to a spherical triangle one can define a *spherical k-gon*. In the proof of Theorem 2 below we will make use of the well-known fact that the sum of the angles of a spherical k-gon is strictly larger than $(k - 2)\pi$.

3 A planar graph that is not a 3-dot product graph

We will construct graphs F, G, H as follows.

(i) We start with K_4, the complete graph on the four vertices t_1, t_2, t_3, t_4.

(ii) To obtain F, we replace each edge t_it_j of K_4 by a path $t_it_{ij}t_{ji}t_j$ of length three.

(iii) The graph F divides the plane into four faces. To obtain G, we place an additional vertex inside each face of F and connect it to all vertices

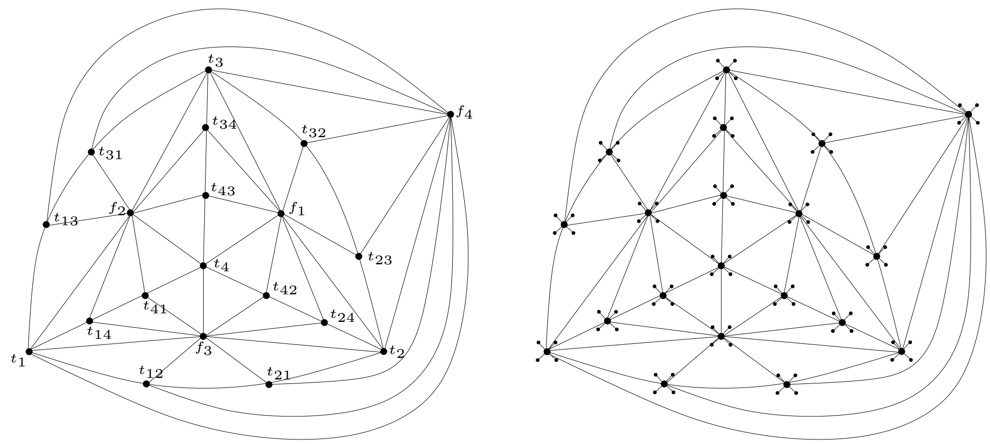


Figure 2: The graphs G (left) and H (right).

on the outer cycle of the face. Here f_i will denote the vertex in the face whose limiting cycle does not contain t_i.

(iv) Finally, to obtain H we attach four leaves to each vertex of G.

The graphs G and H are depicted in Figure 2. For $k \in \mathbb{N}$, let the graph H_k consist of k disjoint copies of H. Clearly H_k is planar for all k. In the rest of this section we shall prove the following.

Theorem 2 *The graph H_k is not a 3-dot product graph for some $k \in \mathbb{N}$.*

Proof of Theorem 2: The proof is by contradiction. Let us assume that H_k has a 3-dot product representation for all k. Then the following must hold.

Claim 3 *For every $\eta > 0$, there is a 3-dot product representation of H, with $\|u(t)\| < 1 + \eta$ for all $t \in V(H)$.*

Proof of Claim 3: Suppose the claim is false. Then there must exist a constant $\eta > 0$ such that in every 3-dot product representation $u : V(H) \to \mathbb{R}^3$ of H, there is a vertex $t \in V(H)$ such that $\|u(t)\| \geq 1 + \eta$.

As H_k is the disjoint union of k copies of H, in any 3-dot product representation $u : V(H_k) \to \mathbb{R}^3$, there is a vertex in each of the k copies of

H whose corresponding vector has length at least $1+\eta$. Let $s_1, \dots, s_k$ denote these vertices. We know $s_i s_j \notin E(H_k)$ for all $1 \le i < j \le k$, and $\|u(s_i)\| \ge 1+\eta$ for all $1 \le i \le k$.

Let us write $v_i := u(s_i)/\|u(s_i)\|$ and $l_i := \|u(s_i)\|$. Since $u(s_i)^T u(s_j) < 1$, we must have

$$v_i^T v_j < \frac{1}{l_i l_j} \le (1+\eta)^{-2}.$$

Write $\rho := \arccos((1+\eta)^{-2})$. Note that $\mathrm{dist}_{S^2}(v_i, v_j) > \rho$ for all $1 \le i < j \le k$. Hence, the spherical caps $\mathrm{cap}(v_1, \rho/2), \dots, \mathrm{cap}(v_k, \rho/2)$ must be disjoint subsets of the sphere. These caps all have the same area, which depends only on ρ, and hence only on η. Let us denote this area by $f(\eta)$. It is possible to express $f(\eta)$ explicitly in terms of η, but there is no need to do this here. We get

$$4\pi = \mathrm{area}(S^2) \ge \sum_{i=1}^{k} \mathrm{area}(\mathrm{cap}(v_i, \rho/2)) = k \cdot f(\eta),$$

which is impossible if we choose $k > 4\pi/f(\eta)$. ■

Let us fix a small η (say $\eta := 1/10^{10}$) and let $u : V(H) \to \mathbb{R}^3$ be the representation provided by Claim 3. For $s \in V(H)$ let us write $l(s) := \|u(s)\|, v(s) := u(s)/\|u(s)\|$. Let us observe that

$$st \in E(H) \quad \text{if and only if} \quad v(s)^T v(t) \ge 1/l(s)l(t).$$

Note also that each spherical arc corresponding to an edge in H has length at most $\rho = \arccos((1+\eta)^{-2})$. Recall that $G \subseteq H$ is the subgraph induced by all non-leaf vertices.

Claim 4 *For every $t \in V(G)$, we have $l(t) \ge 1$.*

Proof of Claim 4: Suppose that some $s \in V(G)$ satisfies $l(s) < 1$. Let s_1, s_2, s_3, s_4 be the four leaves attached to s, in clockwise order. One of the angles $\angle s_1 s s_2$, $\angle s_2 s s_3$, $\angle s_3 s s_4$, $\angle s_4 s s_1$ must be at most $\pi/2$ (they sum to 2π). Without loss of generality, we may assume it is $\gamma := \angle s_1 s s_2$.

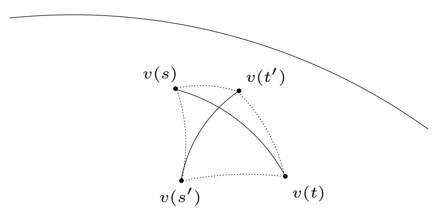
Since $\cos(\gamma) \ge 0$, the spherical cosine rule (2) now implies that

$$\begin{aligned} v(s_1)^T v(s_2) &\ge (v(s)^T v(s_1)) \cdot (v(s)^T v(s_2)) \\ &\ge \left(\frac{1}{l(s)l(s_1)}\right)\left(\frac{1}{l(s)l(s_2)}\right) > \frac{1}{l(s_1)l(s_2)}, \end{aligned}$$

using the property that $l(s) < 1$ for the last inequality. But then we must have $s_1 s_2 \in E(H)$, a contradiction. ■

Claim 5 *Suppose that $st, s't' \in E(G)$ are edges with s, s', t, t' distinct and suppose that the arcs $[v(s), v(t)]$ and $[v(s'), v(t')]$ cross. Then at least one of ss', st', ts', tt' is also an edge of G.*

Proof of Claim 5: Suppose the arcs $[v(s), v(t)]$ and $[v(s'), v(t')]$ cross and $ss', st', ts', tt' \notin E(G)$. Consider the angles of the spherical 4-gon with corners $v(s), v(s'), v(t), v(t')$. Since the sum of the angles of a spherical 4-gon is larger than 2π, at least one angle is larger than $\pi/2$. We may assume without loss of generality that the points are in clockwise order $v(s), v(s'), v(t), v(t')$ and that the angle $\angle v(s')v(s)v(t')$ is more than $\pi/2$.



Then, by version (2) of the spherical cosine rule, we must have

$$v(s')^T v(t') < \left(v(s)^T v(s')\right)\left(v(s)^T v(t')\right).$$

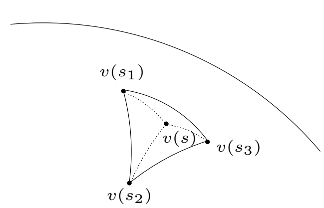
Since $l(s) \geq 1$ by Claim 4 and $ss', st' \notin E(G)$, this gives

$$v(s')^T v(t') < \left(\frac{1}{l(s)l(s')}\right)\left(\frac{1}{l(s)l(t')}\right) \leq \frac{1}{l(s')l(t')},$$

which contradicts $s't' \in E(G)$. ■

Claim 6 *Suppose that s_1, s_2, s_3 form a clique in G, and $v(s)$ lies inside the (smaller of the two areas defined by the) spherical triangle determined by $v(s_1), v(s_2), v(s_3)$. Then either $ss_1 \in E(G)$ or $ss_2 \in E(G)$ or $ss_3 \in E(G)$.*

Proof of Claim 6: At least one of the angles $\angle v(s_1)v(s)v(s_2)$, $\angle v(s_2)v(s)v(s_3)$, $\angle v(s_1)v(s)v(s_3)$ is at least $2\pi/3$. (They sum to 2π.)



Without loss of generality, we may assume it is $\angle v(s_1)v(s)v(s_2)$. The spherical cosine rule (2) gives that

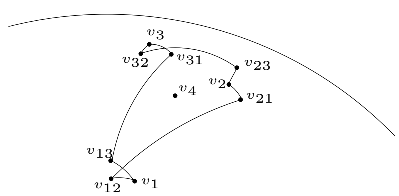
$$v(s_1)^T v(s_2) < \left(v(s)^T v(s_1)\right)\left(v(s)^T v(s_2)\right). \tag{3}$$

If both $ss_1, ss_2 \notin E(G)$, then $v(s)^T v(s_1) \leq 1/(l(s)l(s_1))$ and $v(s)^T v(s_2) \leq 1/(l(s)l(s_2))$. But then, as $l(s) \geq 1$, we get from (3) that $v(s_1)^T v(s_2) < 1/(l(s_1)l(s_2))$. This contradicts that $s_1 s_2 \in E(G)$. ■

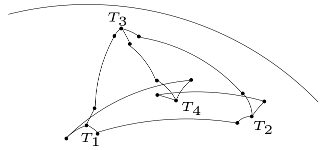
From now on, let us write $v_i = v(t_i)$, $l_i = l(t_i)$ and $v_{ij} = v(t_{ij})$, $l_{ij} = l(t_{ij})$. By Claim 5, the arcs $[v_{ij}, v_{ji}]$ and $[v_{kl}, v_{lk}]$ may not cross (if $\{i,j\} \neq \{k,l\}$). However, the arc $[v_{ij}, v_{ji}]$ could cross an arc of the form $[v_i, v_{ik}]$ or $[v_j, v_{jk}]$.

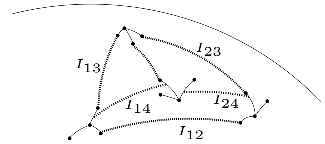
Let C denote the cycle t_1, t_{12}, t_{21}, t_2, t_{23}, t_{32}, t_3, t_{31}, t_{13}, t_1 in G. Let P denote the corresponding spherical polygon. Because each spherical arc corresponding to an edge has length at most ρ, we have $P \subseteq \text{cap}(v_1, 5\rho)$. Also note that $S^2 \setminus P$ consists of at least two path-connected components (by the Jordan curve theorem). As ρ is very small, exactly one of these components has area bigger than 3.9π. We shall refer to this component as the "outside" of P, and the union of the other components, the "inside".

Claim 7 *We may assume without loss of generality that v_4 lies inside the polygon P.*



Proof of Claim 7: For $i \in \{1,2,3,4\}$, let $T_i \subseteq S^2$ denote the union of arcs $\cup_{i \neq j}[v_i, v_{ij}]$. Observe that, by Claim 5, the T_i's are disjoint. Let $I_{ij} \subseteq [v_{ij}, v_{ji}]$ denote the minimal subarc of $[v_{ij}, v_{ji}]$ that connects T_i and T_j. Observe that in this way the I_{ij} are disjoint, I_{ij} hits each of T_i, T_j in exactly one point, and I_{ij} does not intersect T_k for $k \neq i, j$.





A standard argument now shows that, applying a suitable relabelling if necessary, we can assume that T_4 lies inside the smaller of the two regions of $S^2 \setminus (T_1 \cup I_{12} \cup T_2 \cup I_{23} \cup T_3 \cup I_{13})$. But in that case $v_4 \in T_4$ lies inside P as desired. ■

Claim 8 *v_4 lies inside the (smaller of the two areas defined by the) spherical triangle determined by $v(f_4)$ and two consecutive points on P.*

Proof of Claim 8: Notice that, no matter where $v(f_4)$ lies exactly, the great circle through v_4 and $v(f_4)$ hits P twice, and we can speak of the segment of P that lies behind v_4 viewed from $v(f_4)$.

If the vertices of this segment are $v(s_1), v(s_2)$, then the spherical triangle determined by $v(f_4), v(s_1), v(s_2)$ clearly contains $v(t_4)$. ■

Since f_4, s_1, s_2 form a clique in G, Claim 6 together with Claim 8 implies that t_4 must be adjacent to at least one of them. This is a contradiction, since t_4 is neither adjacent to f_4 nor to any vertex of C. This concludes the proof of Theorem 2 ■

We remark that Claims 3, 4, 5 and 6 could be stated more generally. In particular, Claim 3 holds for every graph H such that the graph formed by the disjoint union of k copies of H has a 3-dot product representation for arbitrarily large k. Claim 4 holds for any 3-dot product representation (if one exists) of the graph H obtained from G by appending four leaves to every vertex of G. Claims 5 and 6 hold for every graph G and 3-dot product representation u of G such that $\|u(v)\| \geq 1$ for every $v \in V(G)$.

4 All planar graphs are 4-dot product graphs

The Colin de Verdière parameter $\mu(G)$ of a graph G is the maximum co-rank over all matrices M that satisfy

(i) $M_{ij} < 0$ if $ij \in E(G)$;

(ii) $M_{ij} = 0$ if $ij \notin E(G)$ and $i \neq j$;

(iii) M has exactly one negative eigenvalue; and

(iv) if X is a symmetric $n \times n$ matrix with $X_{ij} = 0$ for all $ij \in E(G)$ and $X_{ii} = 0$ for all i and $MX = 0$, then $X = 0$.

This parameter was introduced by Y. Colin de Verdiére in [2, 3], where it is shown that planar graphs are exactly the graphs G with $\mu(G) \leq 3$.

Kotlov, Lovász and Vempala [5] introduced the following related parameter. Let $\nu(G)$ denote the smallest d such that there exist vectors $u_1, \ldots, u_n \in \mathbb{R}^d$ that satisfy

(i) $u_i^T u_j = 1$ if $ij \in E(G)$;

(ii) $u_i^T u_j < 1$ if $ij \notin E(G)$ and $i \neq j$; and

(iii) if X is a symmetric $n \times n$ matrix such that $X_{ij} = 0$ for all $ij \notin E(G)$ and $X_{ii} = 0$ for all i and $\sum_j X_{ij} u_j = 0$ for all i, then $X = 0$.

Clearly, every graph G is a $\nu(G)$-dot product graph. However, because **(i)** asks for equality and because of the extra demand **(iii)**, G might also be a k-dot product graph for some $k < \nu(G)$. The relation between $\nu(G)$ and $\mu(G)$ is given by the following result.

Theorem 9 ([5]) *If $G \neq K_2$, then $\nu(G) = n - 1 - \mu(\overline{G})$.*

That K_2 is a 4-dot product graph is obvious. That every other planar graph is a 4-dot product graph is a direct consequence of Theorem 9 and the following result.

Theorem 10 ([5]) *If G is the complement of a planar graph, then $\mu(G) \geq n - 5$.*

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