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Cops and Robbers on Geometric Graphs

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Cops and robbers is a turn-based pursuit game played on a graph G. One robber is pursued by a set of cops. In each round, these agents move between vertices along the edges of the graph. The cop number $c(G)$ denotes the minimum number of cops required to catch the robber in finite time. We study the cop number of geometric graphs. For points $x_1, \ldots, x_n \in \mathbb{R}^2$, and $r \in \mathbb{R}^+$, the vertex set of the geometric graph $G(x_1, \ldots, x_n; r)$ is the graph on these n points, with x_i, x_j adjacent when $\|x_i - x_j\| \leq r$. We prove that $c(G) \leq 9$ for any connected geometric graph G in $\mathbb{R}^2$ and we give an example of a connected geometric graph with $c(G) = 3$. We improve on our upper bound for random geometric graphs that are sufficiently dense. Let $\mathcal{G}(n, r)$ denote the probability space of geometric graphs with n vertices chosen uniformly and independently from $[0, 1]^2$. For $G \in \mathcal{G}(n, r)$, we show that with high probability (whp), if $r \geq K_1(\log n/n)^{1/4}$, then $c(G) \leq 2$, and if $r \geq K_2(\log n/n)^{1/5}$, then $c(G) = 1$ where $K_1, K_2 > 0$ are absolute constants. Finally, we provide a lower bound near the connectivity regime of $\mathcal{G}(n, r)$: if $r \leq K_3 \log n/\sqrt{n}$ then $c(G) > 1$ whp, where $K_3 > 0$ is an absolute constant.

1. Introduction

The game of cops and robbers is a full information game played on a graph G. The game was introduced independently by Nowakowski and Winkler [26] and Quilliot [31]. During play, one robber R is pursued by a set of cops $C_1, \ldots, C_\ell$. Initially, the cops choose their locations on the vertex set. Next, the robber chooses his location. The cops and the robber are aware of the location of all agents during play, and the cops can

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coordinate their motion. On the cop turn, each cop moves to an adjacent vertex, or remains stationary. This is followed by the robber turn, and he moves similarly. The game continues with the players alternating turns. The cops win if they can catch the robber in finite time, meaning that some cop is colocated with the robber. The robber wins if he can evade capture indefinitely.

The original formulation [26, 31] concerned a single cop chasing the robber. These papers characterized the structure of *cop-win* graphs for which a single cop has a winning strategy. For $v \in V(G)$, the *neighborhood* of v is $N(v) = \{u \in V(G) \mid (u,v) \in E(G)\}$ and the *closed neighborhood* of v is $\overline{N}(v) = \{v\} \cup N(v)$. When $\overline{N}(u) \subseteq \overline{N}(v)$, we say that u is a *pitfall*. A graph is *dismantlable* if we can reduce G to a single vertex by successively removing pitfalls.

Theorem 1.1 ([26, 31]). *G is dismantlable if and only if $c(G) = 1$.*

Aigner and Fromme [1] introduced the multiple cop variant described above. For a fixed graph G, they defined the *cop number* $c(G)$ as the minimum number of cops for which there is a winning cop strategy on G. Among their results, they proved the following.

Theorem 1.2 ([1]). *If G is a connected planar graph, then $c(G) \leq 3$.*

Various authors have studied the cop number of families of graphs [13, 12, 24, 25]. Recently, significant attention has been directed towards Meyniel's conjecture (found in [12]) that $c(G) = O(\sqrt{n})$ for any n vertex graph. The best current bound is $c(G) \leq n2^{-(1+o(1))\sqrt{\log n}}$, obtained independently in [22, 32, 14]. The history of Meyniel's conjecture is surveyed in [5]. For further results on vertex pursuit games on graphs, see the surveys [3, 17] and the monograph [9].

Herein, we study the game of cops and robbers on geometric graphs in $\mathbb{R}^2$. Given points $x_1, \ldots, x_n \in \mathbb{R}^2$ and $r \in \mathbb{R}^+$, the *geometric graph* $G = G(x_1, \ldots, x_n; r)$ has vertices $V(G) = \{1, \ldots, n\}$ and $ij \in E(G)$ if and only if $\|x_i - x_j\| \leq r$. Geometric graphs are widely used to model ad-hoc wireless networks [16, 34]. For convenience, we will consider $V(G) = \{x_1, \ldots x_n\}$, referring to "point x_i" or "vertex x_i" when this distinction is required. Our first result gives a constant upper bound on the cop number of 2-dimensional geometric graphs.

Theorem 1.3. *If G is a connected geometric graph in $\mathbb{R}^2$, then $c(G) \leq 9$.*

The proof of this theorem is an adaptation of the proof of Theorem 1.2. This adaptation requires three cops on a geometric graph to play the role of a single cop on a planar graph. We also give an example of a geometric graph requiring 3 cops.

Recent years have witnessed significant interest in the study of random graph models, motivated by the need to understand complex real world networks. In this setting, the game of cops and robbers is a simplified model for network security. There are many recent results on cops and robbers on random graph models, including the Erdős-Renyi model and random power law graphs [7, 23, 29, 10, 8, 30]. We add to this list of stochastic models by considering cops and robbers on random geometric graphs. A *random geometric graph* G on $[0,1]^2$ contains of n points drawn uniformly at random. Two points $x, y \in V(G)$ are adjacent when the distance between them is within the connectivity radius, i.e. $\|x-y\| \leq r$. We denote the probability space of random geometric graphs by $\mathcal{G}(n,r)$. Typically, we view the radius as a function $r(n)$, and then study the asymptotic properties of $\mathcal{G}(n,r)$ as n increases. We say that event A occurs *with high probability*, or *whp*, when $\mathbb{P}[A] = 1 - o(1)$ as n tends to infinity, or equivalently, $\lim_{n\to\infty} \mathbb{P}[A] = 1$. For example, $G \in \mathcal{G}(n,r)$ is connected whp if $r = \sqrt{\frac{\log n + \omega(n)}{\pi n}}$. (Here and in the remainder of this paper, $\omega(n)$ denotes an arbitrarily slowly growing function.) For this and further results on $\mathcal{G}(n,r)$, see the monograph [28].

We improve on the bound of Theorem 1.3 when our random geometric graph is sufficiently dense. Essentially, we determine thresholds for which we can successfully adapt known pursuit evasion strategies to the geometric graph setting. Typical analysis of $\mathcal{G}(n,r)$ focusses on the homogeneous aspects of the resulting graph, resulting from tight concentration around the expected structural properties. Our cop strategies rely on these homogeneous aspects.

When studying $G \in \mathcal{G}(n,r)$, it is often productive to tile $[0,1]^2$ into small squares, chosen so that whp, there is a vertex in each square, and vertices in neighboring squares are adjacent in G. We then use the induced grid on these vertices to analyze properties of G, cf. [4, 11]. It is easy to show that the 2-dimensional grid has cop number 2. When our random geometric graph is dense enough, we can adapt a winning two cop strategy on the grid to obtain a winning strategy on $\mathcal{G}(n,r)$.

Theorem 1.4. *There is a constant $K_1 > 0$ such that the following holds. If $G \in \mathcal{G}(n,r)$ on $[0,1]^2$ with $r \geq K_1(\log n/n)^{\frac{1}{4}}$ then $c(G) \leq 2$ whp.*

A further increase in the connectivity radius leads to an even denser geometric graph, so that eventually the cops and robbers game on $\mathcal{G}(n,r)$ becomes quite similar to a turn-based pursuit evasion game on $[0,1]^2$. Such pursuit evasion games on $\mathbb{R}^d$ and in polygonal environments have been well studied, using winning criteria such as capture [33, 20, 6] and line-of-sight visibility [21, 15, 18]. It is known [33, 20] that pursuers can win the capture game in $\mathbb{R}^d$ if and only if the evader starts in the interior of the convex hull of the initial pursuer locations. Furthermore, a single pursuer can always catch the quarry in a bounded region, such as $[0,1]^2$. We use the dismantlable criterion of Theorem 1.1 to prove that a sufficiently dense $\mathcal{G}(n,r)$ also requires a single pursuer.

Theorem 1.5. *There is a constant $K_2 > 0$ such that the following holds. If $G \in \mathcal{G}(n,r)$ on $[0,1]^2$ with $r \geq K_2\,(\log n/n)^{\frac{1}{5}}$, then $c(G) = 1$ whp.*

We note that Theorem 1.5 was proven independently by Alon and Prałat [2] using a graph pursuit algorithm in the spirit of [33, 20].

Finally we also give a lower bound of the cop number of $\mathcal{G}(n,r)$ proving that some random geometric graphs beyond the connectivity threshold require at least two cops. This answers a question of Alon [2].

Theorem 1.6. *There is a constant $K_3 > 0$ such that the following holds. If $G \in \mathcal{G}(n,r)$ on $[0,1]^2$ with $r \le K_3 \log n/\sqrt{n}$, then $c(G) > 1$ whp.*

We do not know whether any of our multiple cop bounds are tight. We are particularly hopeful that the bound for arbitrary geometric graphs can be improved.

2. Notational conventions

We begin by setting some notation. For $x \in \mathbb{R}^2$ and $r \in \mathbb{R}$, define the ball $B(x,r) = \{y \in \mathbb{R}^2 : \|x-y\| \le r\}$.

In the standard formulation of cops and robbers, the cops are first to act in each round. In continuous pursuit evasion games, the evader is usually first to act. The distinction is merely notational, and we choose to view the robber as the first to act in each round. This leads to a more intuitive notation for the game state in our proofs below. Indeed, our cops are always reacting to the robber's previous move (which was made according to some unknown strategy), so it is useful to group these two moves together in a single round.

We formally describe the game of cops and robbers using this notational convention. Before the game begins, the ℓ cops place themselves on the graph at vertices $C_1^0, \dots, C_\ell^0$. Then the game begins. In the first round, the robber chooses his location R^1. Next the cops begin the chase, moving to vertices $C_1^1, \dots, C_\ell^1$ where $C_j^1 \in \overline{N}(C_j^0)$. For $i \ge 2$, the ith round starts in configuration $(R^{i-1}, C_1^{i-1}, \dots, C_\ell^{i-1})$. The robber is first to act, leading to configuration $(R^i, C_1^{i-1}, \dots, C_\ell^{i-1})$ where $R^i \in \overline{N}(R^{i-1})$ at the start of the ith cop turn. Next, the cops move simultaneously to yield configuration $(R^i, C_1^i, \dots, C_\ell^i)$ at the end of the ith round. The cops win if $C_k^i = R^i$ for some finite i,k. Otherwise the robber wins.

Finally, we note that the winning cop criteria has an equivalent formulation. Namely, the cops win if there are finite i,k such that $R^i \in \overline{N}(C_k^{i-1})$. Indeed, C_k would subsequently capture the evader on his ith move, achieving $C_k^i = R^i$. Of course, if $R^i \notin \overline{N}(C_k^{i-1})$ for all k, then the robber cannot be caught in the current round, and his evasion continues.

3. Geometric graphs

In this section, we prove Theorem 1.3. Let $G = G(x_1, \dots x_n; r)$ be a fixed geometric graph. We say that a cop C *controls* a path P if whenever the robber steps onto P, then he steps onto C or is caught by C on his responding move. Let $\operatorname{diam}(G)$ denote the diameter of the graph. Aigner and Fromme [1] prove the following.

Lemma 3.1 ([1]). *Let G be any graph, $u, v \in V(G)$, $u \neq v$ and $P = \{u = v_0, v_1, \dots v_s = v\}$ a shortest path between u and v. A single cop C can control P after at most* $\mathrm{diam}(G)+s$ *moves.*

It takes C at most $\mathrm{diam}(G)$ moves to reach P, and then at most s moves to take control of P. We have the following simple corollary which will be useful for geometric graphs.

Corollary 3.2. *Suppose that there are three cops C_-, C, C_+ chasing robber R on G. Consider a shortest (u, v)-path $P = \{u = v_0, v_1, \dots, v_s = v\}$. After $k \leq \mathrm{diam}(G) + 2s$ moves, the cop C controls P, and $(C_-^k, C^k, C_+^k) = (v_{i-1}, v_i, v_{i+1})$, where we set $v_{-1} = u$ and $v_{s+1} = v$.*

Proof. Start with the three cops colocated on any vertex of P. The cops attain this controlling configuration in two phases. In phase one, cops move as one until they control the path, as in Lemma 3.1. In phase two, C remains in control of the path while C_-, C_+ obtain their proper positions within s moves. Assume that until round $j \geq 1$ of phase two, C_+ is colocated with C. If C stays put on v_i in round j, then C_+ moves to v_{i+1}. If C moves from v_i to v_{i-1} then C_+ stays put on v_i. Otherwise, both C and C_+ move to v_{i+1}. After at most s rounds, C must either stay put or move left, and C_+ attains his proper position. Similarly, C_- attains his position within s rounds. □

Geometric graphs are frequently non-planar. Because of crossing edges, simply keeping R from stepping onto P does not necessarily prevent him from moving from one side of P to the other. We say that R *crosses* P at time t if the closed segment $R^{t-1}R^t$ has nonempty intersection with the closed segments corresponding to the edges of P. The additional guards flanking C ensure that once the three cops are positioned as in Corollary 3.2, R cannot cross P. On a geometric graph, we say that a set of cops *patrols* a path P if they control P and whenever R crosses P, he is caught in the subsequent cop move.

Lemma 3.3. *Let $P = \{v_0, \dots, v_t\}$ be a shortest path on a geometric graph $G(x_1, \dots, x_n; r)$. Suppose that the cops C_-, C, C_+ are located on v_{i-1}, v_i, v_{i+1} respectively, and that cop C controls P. Then these three cops patrol P.*

Proof. If the robber steps onto P then C will capture him. Suppose that the robber can cross P without losing the game, and does so from position R^t to R^{t+1}. We characterize some constraints on the location of R^t. Consider the configurations $(R^t, C_-^{t-1}, C^{t-1}, C_+^{t-1})$ and (R^t, C_-^t, C^t, C_+^t) prior to robber's crossing. These configurations occur in the middle and at the end of round t. At this point, the cops are positioned on three successive vertices of P. We claim that $R^t \notin B(C^t, r)$. Indeed, if $C^{t-1} = C^t$ (so that the cops are stationary in round t), then C can actually catch R at time t, a contradiction. Otherwise $C^t \in \{C_-^{t-1}, C_+^{t-1}\}$, so one of these flanking cops can catch R at time t, also a contradiction.

Next, we observe that the robber cannot be far from the cops. Let $(R^t, C_-^t, C^t, C_+^t) = (R^t, v_{i-1}, v_i, v_{i+1})$. First of all, $R^t \notin B(v_{i-2}, r) \cup B(v_{i+2}, r)$. Indeed, if R^t is close to either of v_{i-2}, v_{i+2} then R could step onto that vertex in round $t+1$ without being caught by C, contradicting the fact that C controls P. Secondly, R^t cannot be within r of any path vertex v_j where $|i-j| > 2$ by a similar argument. We conclude that the robber must cross P between v_{i-2} and v_{i+2}. The region forbidden to R^t along this subpath is shown in Figure 3.1(a).

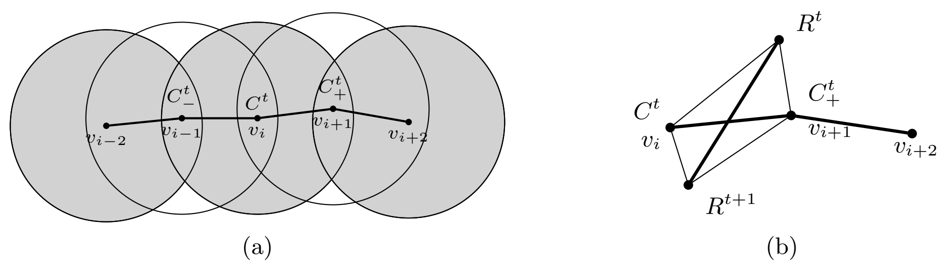


Figure 3.1. (a) The robber must cross between v_{i-2} and v_{i+2}, but R^t cannot lie in the gray region $B(v_{i-2}, r) \cup B(v_i, r) \cup B(v_{i+2}, r)$. (b) The geometry of the quadrilateral $v_i R^t v_{i+1} R^{t+1}$ shows that the robber cannot cross P at edge $v_i v_{i+1}$ without ending in $B(C^t, r) \cup B(C_+^t, r)$.

Without loss of generality, assume that R crosses P so that $R^t R^{t+1}$ intersects $v_i v_{i+1}$ or $v_{i+1} v_{i+2}$. Now $R^{t+1} \notin B(v_i, r) \cup B(v_{i+1}, r)$; otherwise either C or C_+ can immediately catch him. Suppose that $R^t R^{t+1}$ crosses $v_i v_{i+1}$ where $R^t \notin B(v_i, r) \cup B(v_{i+2}, r)$ and $R^{t+1} \notin B(v_i, r) \cup B(v_{i+1}, r)$, as shown in Figure 3.1(b). We have $\|v_i - v_{i+1}\| \leq r$ and $\|R^t - v_i\| > r$. This means that the angle $\measuredangle v_i R^t v_{i+1} < \pi/2$; otherwise in the triangle $v_i v_{i+1} R^t$, this obtuse angle forces $r \geq \|v_i - v_{i+1}\| > \|v_i - R^t\| > r$, a contradiction. Likewise, since $\|R^{t+1} - v_{i+1}\| > r$, we must have $\measuredangle v_i R^{t+1} v_{i+1} < \pi/2$. Therefore $\max\{\measuredangle R^t v_i R^{t+1}, \measuredangle R^t v_{i+1} R^{t+1}\} > \pi/2$, and the resulting obtuse triangle forces $\|R^t - R^{t+1}\| > r$, a contradiction. Therefore R cannot cross P by crossing $v_i v_{i+1}$. An identical argument, replacing v_i with v_{i+2}, shows that R cannot cross $v_{i+1} v_{i+2}$. Therefore, R cannot cross P. □

We now prove that if G is a connected geometric graph in $\mathbb{R}^2$, then $c(G) \leq 9$.

Proof of Theorem 1.3 The proof is a direct adaptation of the Aigner and Fromme [1] proof of Theorem 1.2. In our proof, we need 3 cops to patrol a shortest path of a geometric graph, instead of the single cop required to control a shortest path of a planar graph. The idea of the proof of Aigner and Fromme is divide the pursuit into stages. In stage i, we assign to R a certain subgraph H_i, the *robber territory*, which contains all vertices which R may still safely enter, and to show that, after a finite number of cop-moves, H_i is reduced to $H_{i+1} \subsetneq H_i$. Eventually, there is no safe vertex left for the robber. In each iteration, at most two shortest paths in H_i must be controlled. For a planar graph, this requires one cop per path, and the third cop moves to control another shortest path in H_i. For geometric graphs, Lemma 3.3 shows that 3 cops can patrol any shortest path of a geometric graph. Using that lemma in place of Lemma 3.1, the proof of

Aigner and Fromme for planar graphs with 3 cops becomes a proof for geometric graphs with 9 cops. See [1] for the proof details. □

It is an open question whether this upper bound on the cop number can be improved for the class of geometric graphs. Here we construct a geometric graph that requires 3 cops, which leaves a considerable gap to our upper bound. Aigner and Fromme [1] proved that any graph with minimum degree $\delta(G) \geq 3$ and girth $g(G) \geq 5$ has $c(G) \geq \delta(G)$. We describe a geometric graph G on 1440 vertices with unit connectivity radius which has girth 5 and minimum degree 3, so that $c(G) \geq 3$. A representative subgraph of G appears in Figure 3.2. Start with an annulus having inner radius 55 and outer radius 57. Within the annulus, we create an inner and outer strip of pentagons. Each pentagon corresponds to a one degree angle (or $\pi/180$ radians), so that there are a total of 720 pentagons. We give the vertex locations in polar coordinates $(r : \theta)$ where θ is in degrees. For integral θ, $1 \leq \theta \leq 360$, place a vertex at $(55 : \theta)$ and at $(57 : \theta + 1/2)$. The interior points (separated by 1/2 degree) are chosen in a clockwise repeating pattern $(55 : 2\theta)$, $(56.35 : 2\theta + 0.5)$, $(55.85 : 2\theta + 1)$ and $(56 : 2\theta + 1.5)$ for integral θ, $1 \leq \theta \leq 180$. Simple calculations show that a unit connectivity radius gives the geometric graph as shown in Figure 3.2. For example, the law of cosines calculates the lengths of edges on the outer and inner boundaries as approximately 0.995 and 0.960, respectively.

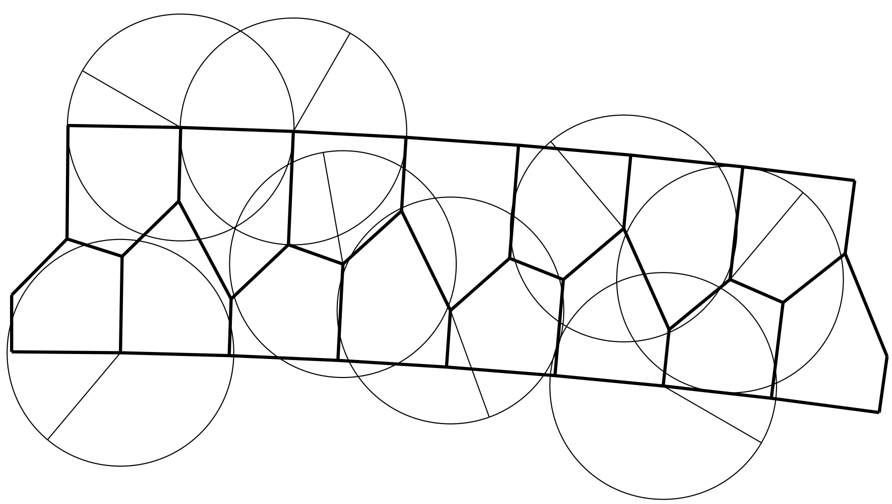


Figure 3.2. Part of a 3-regular geometric graph G on 1440 vertices with $c(G) = 3$. The eight circles show the connectivity neighborhood for each type of vertex.

We must have $c(G) = 3$ since G is planar. Indeed, there is a simple winning strategy for three cops. Have cop C_1 remain stationary on any interior vertex. Place cops C_2, C_3 on vertices on the inner and outer boundaries, separated by half a degree. In each step, one of the boundary cops can take a clockwise step along his boundary while preventing the robber from crossing the shortest path between C_2, C_3. Eventually the robber cannot move counterclockwise because of C_2, C_3, and cannot move clockwise because of C_1.

4. Adapting a grid strategy for $\mathcal{G}(n,r)$

In this section, we prove Theorem 1.4. Our winning two cop strategy is similar to a winning strategy on the grid $P_n \Box P_m$. One cop catches the robber's "shadow" in a copy of P_n, while the other catches the robber's shadow in a copy of P_m. On subsequent moves, either the robber moves towards the boundary, or at least one cop decreases his distance from the robber. Eventually, the robber hits the boundary, and the cops close in for the win. Our cop strategy below follows along similar lines, but accommodates the full range of robber movement.

It is convenient to split the proof of Theorem 1.4 into two parts, a probabilistic part and a deterministic part. Let $V = \{x_1, \dots, x_n\} \subset [0,1]^2$ and let $r \geq s > 0$. Let us say that the tuple $(x_1, \dots, x_n; r, s)$ satisfies condition **(M)** when the following holds:

(M) For every $x \in [0,1]^2$ and every $y \in B(x,r) \cap [0,1]^2$, we have

$$V \cap B(x,r) \cap B(y,s) \neq \emptyset.$$

All the probability theory needed in the proof of Theorem 1.4 is contained in the following lemma.

Lemma 4.1. *Let us set $s := 5\sqrt{\log n/n}$. Let $x_1, \dots, x_n \in [0,1]^2$ be chosen i.i.d. uniformly at random, and let $r \geq s$ be arbitrary. Then $(x_1, \dots, x_n; r, s)$ satisfies condition* **(M)** *whp.*

Proof. Let us set $t := 1/\left\lceil \sqrt{n/2\log n} \right\rceil$. Then $t = (1+o(1))\sqrt{2\log n/n}$ and it is of the form $t = 1/k$ with $k \in \mathbb{N}$ an integer. We can thus tile $[0,1]^2$ into $1/t^2$ squares of dimension $t \times t$. Let Z denote the number of these squares that do not contain any point of $x_1, \dots, x_n$. Then

$$\mathbb{E}[Z] = (1/t^2) \cdot (1-t^2)^n \leq (1/t^2)e^{-nt^2} = (1+o(1))\frac{n}{2\log n} e^{-(1+o(1))2\log n} = o(1).$$

Thus, whp each square contains at least one x_i.

Now let us assume that each square of our dissection indeed contains a point of $x_1, \dots, x_n$ and pick an arbitrary $x \in [0,1]^2$ and $y \in B(x;r) \cap [0,1]^2$. If $\|x-y\| < r - t\sqrt{2}$ then the square of our dissection that contains y is completely contained in $B(x;r)$ (because the diameter of a $t \times t$ square is $t\sqrt{2}$). Hence any point x_i that lies inside this square will clearly do as $\|y - x_i\| \leq t\sqrt{2} < s$. Let us thus assume $r - t\sqrt{2} \leq \|x-y\| \leq r$, and let $z \in [x,y]$ be chosen on the segment between x and y in such a way that $\|z-x\| = r - t\sqrt{2}$. Then the square of our dissection that contains z is contained in $B(x;r)$ and the point x_i inside this square satisfies $\|y - x_i\| \leq \|y-z\| + \|z - x_i\| \leq 2t\sqrt{2} \leq s$. $\square$

Lemma 4.2. *Suppose that $(x_1, \dots, x_n; r, s)$ with $x_1, \dots, x_n \in [0,1]^2$ and $0 < s < r^2/10^{10}$ satisfy condition* **(M)**. *Then $c(G(x_1, \dots, x_n; r)) \leq 2$.*

Proof. We can assume without loss of generality that $r \leq \sqrt{2}$ because otherwise G is a clique and a single cop will be able to catch the robber in a single move. We start by describing the strategy of the cops. The two cops act independently (i.e. the action of C_1 does not depend on the position or movement of C_2 and vice versa). First, we describe only the movements of C_1. Cop C_2 will follow a similar strategy, described below.

We introduce notation for a series of lines and points. Suppose the robber is at point R^t. Let L_1^t be the vertical line through R^t. Let P_1^t denote the point on L_1 exactly $r/3$ below R^t provided this point is above the x-axis. Otherwise P_1^t is the point on the x-axis exactly below R_1^t. Similarly, we define the horizontal line L_2^t and the point P_2^t to the left of R^t on L_2. For simplicity, we occasionally refer to L_1, L_2, P_1, P_2 (without the superscript) to refer to these lines and points with respect to the current position of R.

At time $t = 0$, C_1 starts at a vertex $C_1^0 := x_j$ that is within s of the origin $(0,0)^t$; such an x_j exists because of **(M)**. In each round, the robber will first choose his new location R^{t+1}. The cop then chooses a point $y \in B(C_1^t, r) \cap [0,1]^2$ and finds an $x_i \in B(C_1^t, r) \cap B(y, s)$ (such an x_i exists because of property **(M)**) and chooses as his new location $C_1^{t+1} := x_i$. The strategy of C_1 has three phases:

- S1: Cop C_1 moves right until he reaches a point within s of L_1 and within $r/10^9$ of the x-axis.
- S2: While staying within $r/10^7$ of L_1, cop C_1 moves to within s of the point P_1.
- S3: Cop C_1 tries to stay as close to P_1 as he can.

Stage S1: During stage S1, cop C_1 moves as follows. Let y be the point of $B(C_1^t, r)$ closest to L_1^{t+1}. Then C_1 moves to a point $x_i \in B(C_1^t, r) \cap B(y, s)$. If $y \in L_1$ then stage S1 ends. Otherwise, the cop travels a horizontal distance of at least $r - s$. Thus, stage S1 lasts no more than $\lceil 1/(r-s) \rceil < 10/r$ rounds, since he can keep jumping right by at least $r - s$ and he will reach L_1 before he reaches the right boundary of the unit square (note the cop either starts to the left of L_1 or within s of L_1). Observe that, by the end of stage S1, the y-coordinate of C_1 is at most $s \cdot 10/r < r/10^9$ (as $s < r^2/10^{10}$).

Stage S2: In this stage, the cop will always stay as close to L_1 as he can, and will move closer to his target point P_1 if he can. The round starts with C_1^t within s of L_1^t and within $r/10^9$ of the x-axis. If R^t has y-coordinate smaller than $r/3$ then we are immediately done with stage S2.

If $P_1^{t+1} \in B(C_1^t, r)$ then we can pick an $x_i \in B(C_1^t, r) \cap B(P_1^{t+1}, s)$ and set $C_1^{t+1} := x_i$, thereby ending stage S2. Otherwise, the cop's move depends on how the robber moves. We classify the possible robber moves into four (non-exclusive) types, depending on where the robber jumps, as shown in Figure 4.1. Writing this displacement in polar coordinates $(d : \theta)$, the four types are

- T1: $d \le r/2$.
- T2: $r/2 < d \le r$ and $7\pi/6 \le \theta \le 11\pi/6$.
- T3: $r/2 < d \le r$ and $2\pi/3 \le \theta \le 4\pi/3$.
- T4: $r/2 < d \le r$ and $-\pi/6 \le \theta \le 2\pi/3$.

If R does a T1 move, then we compute $J^{t+1} := R^{t+1} - R^t$. We can write $J^{t+1} = (\ell \cos\alpha, \ell \sin\alpha)$ with $\ell \le r/2$. Assuming C_1^t is within $r/10^7$ of L_1^t, we can move at most

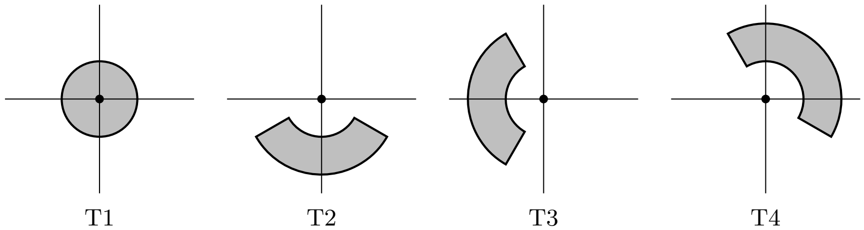


Figure 4.1. The robber move types. In each case the robber will jump into the gray area.

$r(\cos(\alpha)/2 + 1/10^7)$ to the left or right to reach L_1^{t+1}. Thus

$$y := \left(R_x^{t+1}, C_1^t + r\left(1 - \frac{1}{2}\cos\alpha - 1/10^7\right)\right)^T \in L_1^{t+1} \cap B(C_1^t, r),$$

where R_x^{t+1} is the x-coordinate of R^{t+1}. We pick $x_i \in B(C_1^{t+1}, r) \cap B(y, s)$ and set $C_1^{t+1} := x_i$. Observe x_i is within s of L_1^{t+1} and that the distance between C_1 and R has decreased by at least $r\left(1 - \frac{1}{2}\sin\alpha - \frac{1}{2}\cos\alpha - 1/10^7\right) - s \geq r\left(1 - \frac{1}{2}\sqrt{2} - 1/10^7 - 1/10^{10}\right) > r/4$.

If R does a T2 move, then L_1 moves left or right by at most $r\cos(\pi/6) = \sqrt{3}r/2$ and R moves down by at least $r\sin(\pi/6) = r/2$. Assuming that C_1^t is within $r/10^7$ of L_1^t, we can thus move sideways by at most $(\sqrt{3}/2 + 1/10^7)r$ and reach L_1^{t+1}. We can therefore pick a point $y \in L_1^{t+1} \cap B(C_1^t, r)$ that is at least $(\frac{3}{2} - \sqrt{3}/2 - 1/10^7)r - s > r/2$ closer to R^{t+1} than C_1^t is to R^t. Again we pick $x_i \in B(C_1^{t+1}, r) \cap B(y, s)$ and set $C_1^{t+1} := x_i$.

If R does a T3 or T4 move then we compute $y := R^{t+1} - R^t + C_1^t$, (if $y \notin [0,1]^2$ then we take the point y' on $\partial[0,1]^2$ with minimum distance to y) we pick $x_i \in B(C_1^t, r) \cap B(y, s)$ and we set $C_1^{t+1} := x_i$. Note that this way the distance of C_1 to P_1 cannot increase by more than s.

Stage S3: At present it is not yet clear whether stage S2 will ever finish (and also we may not be able to stay within $r/10^7$ of L_1 indefinitely). If however we do get to stage S3, we observe that R cannot make a T1 or T2 move without getting caught by the cop immediately (see Figure 4.2). Therefore, during stage S3, we act exactly as in the case of stage S2 where R does a T3 or T4 move. This concludes the description of the first cop's movements.

Suppose that during the first $T = 1000/r$ moves of the game the robber does not get caught. Stage S1 will have finished after at most $10/r$ moves. Since $s \cdot T < r/10^7$, we will be able to stay within $r/10^7$ of L_1 for the remaining moves until T, and assuming we reach stage S3 at some time $t < T$ we will be able to stay within $r/10^7$ of P_1 for the remaining moves until T. Thus stage S2 will have finished as soon as we have done at most $14/r$ moves of type T1 or T2 (the first $10/r$ may occur during stage S1 and after that we move closer to P_1 by at least $r/4$ in each T1 or T2 move). Thus, out of the first T moves, at most $14/r$ robber moves are of type T1 or T2.

Completely analogously we can define a strategy for the second cop C_2 that will ensure that in the first T moves no more than $14/r$ robber moves are of type T1 or T3. Cop C_2 tries to attain position on the horizontal line L_2 through R. The stages of his strategy are:

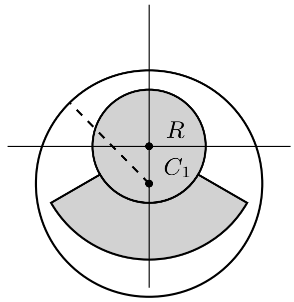


Figure 4.2. If C_1 is within $r/10^7$ of the point $r/3$ below R, then R can no longer make T1 or T2 moves.

S′1: Cop C_2 moves up until he reaches a point within s of L_2 and within $r/10^9$ of the y-axis.
S′2: While staying within $r/10^7$ of L_2, cop C_2 moves to within s of the point P_2.
S′3: Cop C_2 tries to stay as close to P_2 as he can.

Observe that whenever R does a T4 move, then the sum of his coordinates increases by at least

$$\min_{-\pi/6 \le \theta \le \frac{2\pi}{3}} (\sin\theta + \cos\theta)\frac{r}{2} = \left(\frac{\sqrt{3}-1}{4}\right) r.$$

Meanwhile, if the robber makes a $T1$, $T2$ or $T3$ move, the sum of his coordinates decreases by at most $r\sqrt{2}$ (achieved at $\theta = 5\pi/4$). Hence, if the robber did not get caught in the first T moves, then the sum of robbers coordinates at time T is at least

$$\begin{aligned} R_x^T + R_y^T &\ge (T - 28/r) \cdot \left(\tfrac{\sqrt{3}-1}{4}\right) r - (28/r) \cdot r\sqrt{2} \\ &= 972\left(\tfrac{\sqrt{3}-1}{4}\right) - 28\sqrt{2} > 2. \end{aligned}$$

But this is impossible, since the robber stays inside the unit square. It follows that R gets caught by the cops within the first T moves. $\square$

Proof of Theorem 1.4 Follows from Lemmas 4.1 and 4.2 by taking $K_1 = 3 \cdot 10^5$. $\square$

5. A dismantlable $\mathcal{G}(n, r)$

In this section, we prove Theorem 1.5 by showing that when $r \ge K_2(\log n/n)^{1/5}$ the random geometric graph is dismantlable. We begin by setting some notation. Let $c := (\frac{1}{2}, \frac{1}{2})$ denote the center of the unit square $[0,1]^2$. Let us write

$$\mathcal{N}_c(i) := \{1 \le j \le n : \|x_i - x_j\| \le r, \text{ and } \|x_j - c\| < \|x_i - c\|\}.$$

In other words, $\mathcal{N}_c(i)$ is the set of (indices) of vertices adjacent to x_i and closer to the center c than x_i. We will prove the following lemma.

Lemma 5.1. *There is a constant $K_2 > 0$ such that the following holds. Suppose $r \ge K_2(\log n/n)^{1/5}$. Whp the following holds for all $1 \le i \le n$: either $\|x_i - c\| < r/2$, or there is a $j \in \mathcal{N}_c(i)$ such that $\mathcal{N}_c(i) \subseteq \mathcal{N}_c(j)$.*

Assuming that Lemma 5.1 holds, the proof of Theorem 1.5 is straightforward dismantling of the random geometric graph.

Theorem 1.5 We can induce a strict ordering of the vertices according to their distance from the center c, in descending order. Indeed, for any vertices x, y, $\mathbb{P}(\|x-c\| = \|y-c\|) = 0$. By Lemma 5.1, the outermost vertex is a pitfall, and can be removed. We continue to remove vertices until the remaining vertices lie in $B(c, r/2)$. The graph induced by these remaining vertices forms a clique, which is dismantlable. By Theorem 1.1, the graph has $c(G) = 1$. $\square$

The remainder of this section is devoted to proving Lemma 5.1, which requires a series of intermediate geometric lemmas. For $x, y \in \mathbb{R}^2$, let us write

$$W(x, y; r) := \{z \in \mathbb{R}^2 : B(z, r) \supseteq B(x, r) \cap B(y, \|x - y\|)\}. \tag{5.1}$$

Let $[x, y]$ denote the line segment between these two points. Note that

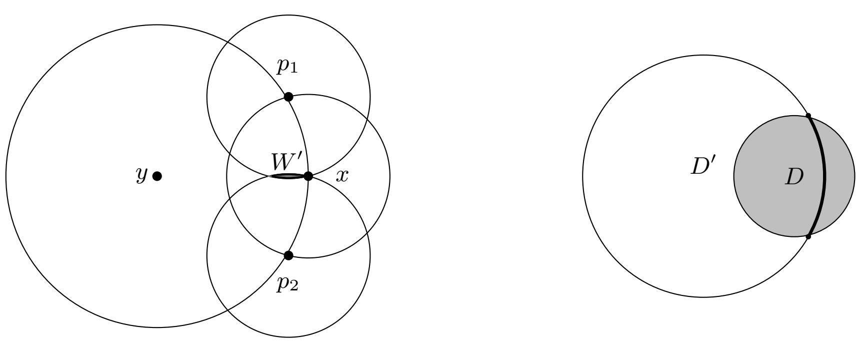
$$\text{if } z \in [x, y] \text{ then } W(x, y; r) \subseteq W(x, z; r). \tag{5.2}$$

Indeed, we have $B(x, r) \cap B(z, \|x - z\|) \subseteq B(x, r) \cap B(y, \|x - y\|)$ so that $W(x, y; r) \supseteq W(x, z; r)$. Observe that $\mathrm{area}(W(x, y; r))$ does not depend on the exact locations of x, y, but only on $\|x-y\|$ and r. We can thus denote $A(d, r) := \mathrm{area}(W(x, y; r))$ for an arbitrary pair x, y with $\|x - y\| = d$. By observation (5.2), the area $A(d, r)$ is nonincreasing in d for a fixed r.

We give a simpler geometric characterization of $W(x, y; r)$ when $\|x - y\| = d > r$. Let p_1, p_2 denote the two intersection points of $\partial B(x, r)$ and $\partial B(y, d)$. Denote

$$W'(x, y; r) := B(p_1, r) \cap B(p_2, r),$$

as shown in Figure 5.1(a).



(a) (b)

Figure 5.1. (a) The set $W' = W'(x, y; r)$. (b) When closed discs intersect, the smaller disc D contains the shortest arc on the bigger disc D' between the intersection points of the boundaries.

Lemma 5.2. *If $\|x - y\| = d > r$ then $W'(x, y; r) = W(x, y; r)$.*

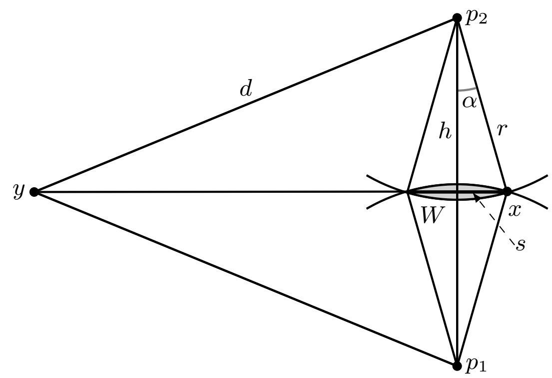


Figure 5.2. Determining the area of $W = W(x, y; r)$.

Proof. Pick any $z \in W(x, y; r)$. We must have $p_1, p_2 \in B(z, r)$, which means that $z \in B(p_1, r) \cap B(p_2, r)$ Therefore $W(x, y; r) \subseteq W'(x, y; r)$.

Picking any $z \in W'(x, y; r)$, we have $p_1, p_2 \in B(z, r)$. Observe that if a closed disc D intersects a disc D' of the same or larger radius then D contains the shortest circular arc along $\partial D'$ between the two intersection points of ∂D and $\partial D'$, see Figure 5.1(b). So $B(z, r)$ contains the part of $\partial B(x, r)$ between p_1 and p_2 that lies inside $B(y, d)$. Using that $d > r$, $B(z, r)$ also contains the part of $\partial B(y, d)$ between p_1 and p_2 that falls inside $B(x, r)$. Thus $B(z, r)$ contains $\partial\left(B(x, r) \cap B(y, \|x - y\|)\right)$. Because both $B(z, r)$ and $B(x, r) \cap B(y, \|x - y\|)$ are convex, it now also follows that $B(x, r) \cap B(y, d) \subseteq B(z, r)$. This shows that $W'(x, y; r) \subseteq W(x, y; r)$. $\square$

We now compute a lower bound for $A(d, r)$ for distant vertices x, y.

Lemma 5.3. *If $d = K \cdot \max\left(r, 1/\sqrt{2}\right)$ where $K > 1$ is a sufficiently large constant, then $A(d, r) = \Omega(r^5)$.*

Proof. Choose $x, y \in \mathbb{R}^2$ with $\|x - y\| = d$. The geometry of $W = W(x, y, r)$ is shown in Figure 5.2. We have

$$\begin{aligned} \operatorname{area}(W) &= 4\left(\pi r^2 \left(\frac{\alpha}{2\pi}\right) - \tfrac{1}{2} r^2 \cos(\alpha)\sin(\alpha)\right) \\ &= r^2 \cdot (2\alpha - \sin(2\alpha)). \end{aligned} \tag{5.3}$$

Indeed, the expression $\pi r^2\left(\frac{\alpha}{2\pi}\right)$ equals the area of a slice of opening angle α out of a disc of radius r, and the term $\frac{1}{2} r^2 \cos(\alpha)\sin(\alpha)$ equals the area of a triangle with sides $h = r\cos(\alpha)$ and $s = r\sin(\alpha)$. Also note that $d^2 = h^2 + (d - s)^2$ and $r^2 = h^2 + s^2$, giving

$$s = r^2/2d = \min\left(\frac{r^2}{K\sqrt{2}}, \frac{r}{2K}\right) = \Omega(r^2).$$

Thus, $\sin(\alpha) = s/r = \Omega(r)$, and because $\sin(x) = x + O(x^3)$, this also gives $\alpha = \Omega(r)$. The approximation $x - \sin(x) = x^3/6 + O(x^5)$, together with (5.3), proves the lemma. $\square$

Our next lemma places a lower bound on area$(W(x, c; r))$ where $c = (\frac{1}{2}, \frac{1}{2})$ is the center of the unit square.

Lemma 5.4. *For all $x \in [0,1]^2$ with $\|x - c\| \geq r/2$, we have*

$$\text{area}\left(W(x, c; r) \cap [0,1]^2 \cap B(c, \|x - c\|)\right) = \Omega(r^5).$$

Proof. Pick the point $\tilde{c}$ on the line L containing x and c, so that $c \in [\tilde{c}, x]$ and $\|x - \tilde{c}\| = d = K \cdot \max(r, 1/\sqrt{2})$, see Figure 5.3. By equation (5.2), $W(x, \tilde{c}; r) \subseteq W(x, c; r)$. Provided

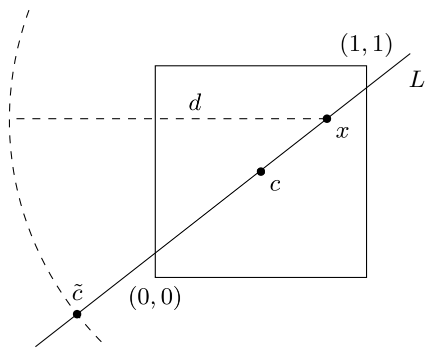


Figure 5.3. Choosing $\tilde{c}$ such that $c \in [\tilde{c}, x]$.

that K is sufficiently large, we have diam$(W(x, \tilde{c}; r)) < r/10^{10}$. Furthermore, both the angle between $\partial B(p_1, r)$ and the line L at their intersection points, and the angle between $\partial B(p_2, r)$ and the line L at their intersection points will be less than 1 degree. It follows directly that $W(x, \tilde{c}; r) \subseteq [0,1]^2 \cap B(c, \|x - c\|)$ for every $x \in [0,1]^2 \setminus B(c, r/2)$. Applying Lemma 5.3 completes the proof. $\square$

We conclude this section with the proof of our main lemma: that for every vertex x_i such that $\|x_i - c\| > r/2$, there is a $j \in \mathcal{N}_c(i)$ such that $\mathcal{N}_c(i) \subseteq \mathcal{N}_c(j)$.

Proof of Lemma 5.1 We can assume without loss of generality that $r \leq \sqrt{2}$ (otherwise $\|x_i - c\| < r/2$ holds trivially for all i). Let Z denote the number of indices i such that $\|x_i - c\| \geq r/2$ and there is no $j \in \mathcal{N}_c(i)$ such that $\mathcal{N}_c(j) \supseteq \mathcal{N}_c(i)$. Then $\mathbb{E}Z$ can be bounded above by:

$$\begin{aligned}
\mathbb{E}[Z] &\leq n \int_{[0,1]^2 \setminus B(c, r/2)} \left(1 - \text{area}(W(x, c; r) \cap [0,1]^2)\right)^{n-1} \mathrm{d}x \\
&\leq n \left(1 - \Omega(r^5)\right)^{n-1} \leq n \exp\left[-\Omega(nr^5)\right]
\end{aligned}$$

Thus, if we chose K_2 sufficiently large we have $\mathbb{E}Z \leq \exp[\log n - \Omega(nr^5)] = \exp[-\Omega(\log n)] = o(1)$. So the assertion of the lemma holds whp. $\square$

6. $\mathcal{G}(n,r)$ near the connectivity threshold is not cop-win

In this section, we prove that some random geometric graphs require at least two cops. In particular, when we are near the connectivity threshold, the graph is not dismantlable whp.

Proof of Theorem 1.6 Without loss of generality we can assume $r \geq \frac{1}{2}\sqrt{\log n/n}$, because by a result of Penrose [27] our graph is disconnected whp for smaller choices of r (obviously a disconnected graph is not cop-win). We will show that there is a small constant $K_3 > 0$ such that if $r \leq K_3 \log n/\sqrt{n}$ then whp the graph is not dismantlable.

Intuitively, we are hunting for a subset of $[0,1]^2$ as shown in Figure 6.1. Start with an N-gon with side length ρ_1, slightly smaller than r. Draw a small disc $B(c_i, \rho_2)$ around each corner, where $\rho_1 + 2\rho_2 = r$. We want each disc $B(c_i, \rho_2)$ to contain exactly one vertex of G, say x_i. Next, we consider the sets $B(x_{i-1}, r) \cap B(x_{i+1}, r)$. We want this intersection to contain no other vertices besides x_i. If we can find such a structure, it creates a cycle $\{x_1, \ldots x_N\}$ in G such that x_i the only vertex in G that is adjacent to both x_{i-1}, x_{i+1} (addition modulo N). Therefore G is not dismantlable because none of the x_i will ever become pitfalls.

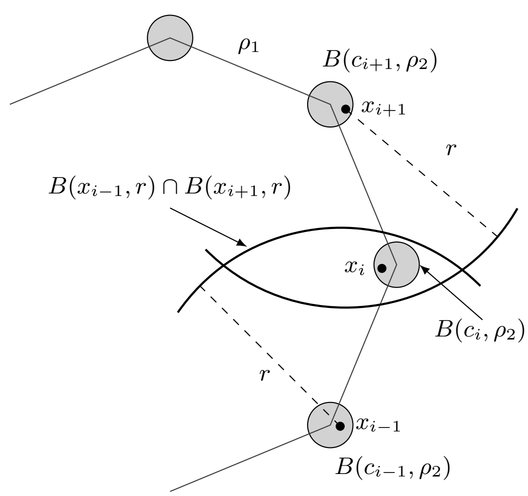


Figure 6.1. For an N-gon with side length ρ_1, we want each $B(c_i, \rho_2)$ to contain a single vertex, and we want each $B(x_{i-1}, r) \cap B(x_{i+1}, r)$ to contain no additional vertices.

We now prove the existence of such a structure. Let N denote the number of vertices of the cycle; we will specify this value later. Set $\rho_1 = r - r/N^2$ and $\rho_2 = r/2N^2$. Consider a regular N-gon $\Gamma \subseteq [0,1]^2$, whose edges each have length ρ_1. (Once we fix our choice of N, we shall see later that Γ fits easily inside the unit square $[0,1]^2$.) Let us label the corners of Γ as $c_0, \ldots, c_{N-1}$ for convenience, where of course c_i is next to c_{i-1} and c_{i+1} (addition of indices modulo N). We will insist that, for each $0 \leq i \leq N-1$ there is a point $x_{j_i} \in B(c_i, \rho_2)$ with

$$\{x_1, \dots, x_n\} \cap B(c_i, \rho_2) = \{x_{j_i}\}, \tag{6.1}$$

and the point x_{j_i} is also the unique common neighbor of the two points $x_{j_{i-1}}$ and $x_{j_{i+1}}$, i.e.

$$\{x_1, \dots, x_n\} \cap B(x_{j_{i-1}}, r) \cap B(x_{j_{i+1}}, r) = \{x_{j_i}\}. \tag{6.2}$$

Observe that

$$\begin{aligned}\|c_{i+1} - c_{i-1}\| &= 2\rho_1 \sin\left(\frac{\pi(N-2)}{2N}\right) \;=\; 2\rho_1 \cos\left(\frac{\pi}{N}\right) \\ &= 2r\left(1 - 1/N^2\right)\left(1 - O\left(1/N^2\right)\right) \;=\; 2r - O\left(r/N^2\right)\end{aligned}$$

using the Taylor approximation $\cos(x) = 1 - \frac{1}{2}x^2 + O(x^4)$. Hence for any $x \in B(c_{i+1}, \rho_2)$ and $y \in B(c_{i-1}, \rho_2)$ we also have $\|x - y\| = 2r - O(r/N^2)$. Let us write $W(x,y) := B(x,r) \cap B(y,r)$. By the same computation as equation (5.3),

$$\operatorname{area}(W(x,y)) = r^2(2\beta - \sin(2\beta)) = O(r^2\beta^3),$$

where β is a small angle with $\cos\beta = \frac{1}{2}\|x-y\|/r = 1 - O(1/N^2)$, so that $\beta = O(1/N)$ (again using the Taylor expansion of cosine), Hence

$$\operatorname{area}(W(x,y)) = O(r^2/N^3). \tag{6.3}$$

Rather than computing directly in the standard random geometric graph, it helps to consider a "Poissonized" version. Consider an infinite sequence $x_1, x_2, \dots$ of random points, i.i.d. uniformly at random on the unit square. The ordinary random geometric graph, which we will denote by G_O for the rest of the proof, is just $G(x_1, \dots, x_n; r)$. Now let $Z \overset{d}{=} \mathrm{Po}(n)$ be a Poisson random variable of mean n, independent of the points $x_1, x_2, \dots$ and consider the random geometric graph $G(x_1, \dots, x_Z; r)$ on the points $x_1, \dots, x_Z$ which we will denote by G_P. Observe that the points $x_1, \dots, x_Z$ constitute a Poisson process of intensity n on the unit square, which has the convenient properties that for every $A \subseteq [0,1]^2$ the number of points that fall in A is a Poisson random variable with mean $n \cdot \operatorname{area}(A)$, and that for any two disjoint sets A, B the number of points in A is independent of the number of points in B (cf. [19]). This makes G_P slightly easier to handle than G_O. We shall first do our probabilistic computations for the Poissonized version G_P and then we'll derive the results for the original model G_O from those for the Poissonized one.

Let us say the polygon Γ is *good* if it satisfies the demands of equations (6.1) and (6.2) with Z swapped for n. Employing the useful independence properties of the Poisson process we now see that

$$\begin{aligned}\mathbb{P}[\Gamma \text{ is good}] &= \left(\mathbb{P}[\mathrm{Po}((n\pi r^2/4N^4) = 1])\right)^N \cdot \mathbb{P}[\mathrm{Po}(n \cdot O(r^2/N^2)) = 0] \\ &= \left((n\pi r^2/4N^4)\exp(-n\pi r^2/4N^4)\right)^N \cdot \exp(-O(nr^2/N^2)) \\ &= \exp\left(N\log(n\pi r^2/4) - 4N\log N - O(nr^2/N^2)\right).\end{aligned}$$

Considering the right hand side of the first inequality, the first term is the probability that the N discs $B(c_i, \rho_2)$ contain exactly one random point, and the second term is the probability that the N sets $(B(x_{i-1},r)\cap B(x_{i+1},r))\backslash B(c_i,\rho_2)$ contain no random points. We now choose $N = \lceil (n\pi r^2)^{1/4} \rceil$ and choose $K_3 > 0$ to be small enough so that we obtain

$$\mathbb{P}(\Gamma \text{ is good}) \geq \exp\left(-O\left(\sqrt{nr^2}\right)\right) \geq \exp\left(-\frac{1}{2}\log n\right) = n^{-\frac{1}{2}}$$

because $r \leq K_3 \log n/\sqrt{n}$ by assumption. Also note that as promised before, the polygon Γ fits easily inside the unit square as it has diameter $O(rN) = O(r(nr^2)^{1/4}) = o(1)$.

Let us now place shifted copies $\Gamma_1,\dots,\Gamma_M$ of Γ inside the unit square in such a way that they are contained in $[0,1]^2$ and their centers are separated by at least $10\,\mathrm{diam}(\Gamma) = \Theta(rN) = \Theta(n^{1/4}r^{3/2}) = n^{-1/2+o(1)}$. (Recall we assumed without loss of generality that $r = \Omega(\sqrt{\log n/n})$.) Then we can place $M = \Omega((1/rN)^2) = n^{1-o(1)}$ such shifted copies, with their centers forming a lattice in $[0,1]^2$. Let X denote the number of Γ_is that are good. Now notice that the events that the Γ_i are good are independent of each other as they concern disjoint areas of the plane. Hence X is distributed like a binomial with parameters $M = n^{1-o(1)}$ and $p \geq n^{-\frac{1}{2}}$. Thus:

$$\mathbb{P}[X=0] = (1-p)^M \leq e^{-Mp} \leq e^{-n^{1/2-o(1)}} = o(1).$$

So $X > 0$ whp.

Consider the original random geometric graph G_O again. Let X_P denote the number of good Γ_is under the Poisson model, and let X_O denote the number of good Γ_is under the original model. We have, with $K > 0$ an arbitrary constant:

$$\begin{array}{rcl} \mathbb{P}[X_O = 0|X_P > 0] & = & \sum_{z=0}^{\infty} \mathbb{P}[X_O = 0|X_P > 0, Z = z]\mathbb{P}[X_P > 0|Z = z]\mathbb{P}[Z = z] \\ & \leq & \sum_{z=0}^{\infty} \mathbb{P}[X_O = 0|X_P > 0, Z = z]\mathbb{P}[Z = z] \\ & \leq & \sum_{z=n-K\sqrt{n}}^{n+K\sqrt{n}} \mathbb{P}[X_O = 0|X_P > 0, Z = z]\mathbb{P}[Z = z] \\ & & +\mathbb{P}[|Z-n| > K\sqrt{n}]. \end{array} \tag{6.4}$$

By Chebyschev's inequality we have

$$\mathbb{P}\left[|Z-n| > K\sqrt{n}\right] \leq \mathrm{Var}(Z)/(K\sqrt{n})^2 = 1/K^2.$$

Now consider the term $\mathbb{P}[X_O = 0|X_P > 0, Z = z]$. If $z = n$ then it clearly equals 0. Let us take $n - K\sqrt{n} \leq z < n$. If we condition on the event that $X_P > 0, Z = z$, then we can fix a good Γ_i, say with "corners" $(x_{i_1},\dots,x_{i_N})$. If $X_O = 0$ then the set $A := \bigcup_{j=1}^{N} W(x_{i_{j-1}}, x_{i_{j+1}})$ must contain one of the points $x_{z+1},\dots,x_n$. By equation (6.3), $\mathrm{area}(A) = N\cdot O(r^2/N^3) = O(r^2/N^2)$. Thus, for $n - K\sqrt{n} < z < n$ we have

$$\begin{array}{rcl} \mathbb{P}[X_O = 0|X_P > 0, Z = z] & \leq & (n-z)\cdot O(r^2/N^2) \\ & \leq & K\sqrt{n}\cdot O(r/\sqrt{n}) \\ & = & o(1), \end{array}$$

using $N = \lceil (\pi n r^2)^{1/4} \rceil$. Observe that the $o(1)$ bound is *uniform* over all $n - K\sqrt{n} < z < n$.

Similarly, if we condition on the event that $X_P > 0, Z = z$ with $n < z \leq n + K\sqrt{n}$, we can pick an N-tuple $(x_{i_1},\dots,x_{i_N})$ uniformly at random from all N-tuples that are "corners" of a good Γ_i. The indices $i_1,\dots,i_N$ are a uniformly random sample (without replacement) from $\{1,\dots,z\}$. Now, if $X_O = 0$, it must hold that one of $i_1, \dots i_N$ is larger than n. Note $\mathbb{P}(i_j > n) = (z-n)/z$ for $j = 1,\dots,N$, and so

$$\mathbb{P}[X_O = 0 | X_P > 0, Z = z] \le N\left(\frac{z-n}{z}\right) \le (\pi n r^2)^{\frac{1}{4}}\left(\frac{K\sqrt{n}}{n}\right) = K\pi^{\frac{1}{4}} n^{-\frac{1}{4}} r^{\frac{1}{2}} = o(1).$$

Observe that again the $o(1)$ bound is *uniform* over all z considered. Combining these bounds with (6.4) we get

$$\begin{aligned}\mathbb{P}[X_O = 0 | X_P > 0] &\le 1/K^2 + \textstyle\sum_{z=n-K\sqrt{n}}^{n+K\sqrt{n}} o(1)\cdot\mathbb{P}[Z = z]\\ &= 1/K^2 + o(1).\end{aligned}$$

By sending $K \to \infty$, we see that $\mathbb{P}[X_O = 0|X_P > 0] = o(1)$, so

$$\mathbb{P}[X_O > 0] \ge \mathbb{P}[X_O > 0|X_P > 0]\mathbb{P}[X_P > 0] = (1-o(1))(1-o(1)) = 1-o(1),$$

which concludes the proof. □

References

[1] M. Aigner and M. Fromme, *A game of cops and robbers*, Discrete Appl. Math. **8** (1983), 1–12.
[2] N. Alon, personal communication, 2011.
[3] B. Alspach, *Sweeping and searching in graphs: a brief survey*, Mathematiche **59** (2006), 5–37.
[4] C. Avin and G. Ercal, *On the cover time and mixing time of random geometric graphs*, Theor. Comput. Sci. **380** (2007), no. 1-2, 2–22.
[5] W. Baird and A. Bonato, *Meyniel's conjecture on the cop number: a survey*, Submitted.
[6] D. Bhadauria and V. Isler, *Capturing an evader in a polygonal environment with obstacles*, 22nd International Joint Conference on Artificial Intelligence, 2011, To appear.
[7] B. Bollobás, G. Kun, and I. Leader, *Cops and robbers in random graphs*, Submitted.
[8] A. Bonato, G. Kemkes, and P. Prałat, *Almost all cop-win graphs contain a universal vertex*, Submitted.
[9] A. Bonato and R. Nowakowski, *The game of cops and robbers on graphs*, American Mathematical Society, 2011.
[10] A. Bonato, P. Prałat, and C. Wang, *Pursuit-evasion in models of complex networks*, Internet Mathematics **4** (2007), no. 4, 419–436.
[11] C. Cooper and A. Frieze, *The cover time of random geometric graphs*, Random Structures and Algorithms **38** (2011), 324–349.
[12] P. Frankl, *Cops and robbers in graphs with large girth and Cayley graphs*, Discrete Applied Mathematics **17** (1987), 301–305.
[13] ——, *On a pursuit game on Cayley graphs*, Combinatorica **7** (1987), no. 1, 67–70.
[14] A. Frieze, M. Krivelevich, and P.-S. Lo, *Variations on cops and robbers*, J. Graph Theory, To appear.
[15] L. J. Guibas, J.-C. Latombe, S. M. Lavalle, D. Lin, and R. Motwani, *A visibility-based pursuit-evasion problem*, International Journal of Computational Geometry and Applications **9** (1996), 471–494.
[16] P. Gupta and P. R. Kumar, *Critical power for asymptotic connectivity in wireless netowrks*, Stochastic Analysis, Control, Optimization and Applications (Boston), Birkhaüser, 1998.
[17] G. Hahn, *Cops, robbers and graphs*, Tatra Mountain Mathematical Publications **36** (2007), 163–176.

[18] V. Isler, S. Kannan, and S. Khanna, *Randomized pursuit-evasion in a polygonal environment*, IEEE Transactions on Robotics **5** (2005), no. 21, 864–875.

[19] J. Kingman, *Poisson processes*, Oxford University Press, Oxford, 1993.

[20] S. Kopparty and C. V. Ravishankar, *A framework for pursuit evasion games in* $\mathbb{R}^n$, Inf. Process. Lett. **96** (2005), 114–122.

[21] S. M. Lavalle, D. Lin, L. J. Guibas, J.-C. Latombe, and R. Motwani, *Finding an unpredictable target in a workspace with obstacles*, Proceedings of the 1997 IEEE International Conference on Robotics and Automation, 1997, pp. 737–742.

[22] L. Lu and X. Peng, *On Meyniel's conjecture of the cop number*, Submitted.

[23] T. Łuczak and P. Prałat, *Chasing robbers on random gaphs: zigzag theorem*, Random Structures and Algorithms **37** (2010), 516–524.

[24] A. Mehrabian, *The capture time of grids*, Discrete Mathematics **311** (2011), no. 1, 102 – 105.

[25] S. Neufeld and R. Nowakowski, *A game of cops and robbers played on products of graphs*, Discrete Mathematics **186** (1998), no. 1-3, 253 – 268.

[26] R. Nowakowski and P. Winkler, *Vertex-to-vertex pursuit in a graph*, Discrete Mathematics **43** (1983), no. 2–3, 235 – 239.

[27] M. D. Penrose, *The longest edge of the random minimal spanning tree*, Ann. Appl. Probab. **7** (1997), no. 2, 340–361.

[28] M. D. Penrose, *Random geometric graphs*, Oxford University Press, 2003.

[29] P. Prałat, *When does a random graph have constant cop number?*, Australasian Journal of Combinatorics **46** (2010), 285–296.

[30] P. Prałat and N. Wormald, *Meyniel's conjecture holds in random graphs*, Submitted.

[31] A. Quilliot, *Jeux et pointes fixes sur les graphes*, Ph.D. thesis, Université de Paris VI, 1978.

[32] A. Scott and B. Sudakov, *A bound for the cops and robbers problem*, SIAM J. Discrete Math., To appear.

[33] J. Sgall, *A solution to David Gale's lion and man problem*, Theoretical Comp. Sci. **259** (2001), no. 1–2, 663–670.

[34] F. Xue and P. R. Kumar, *Scaling laws for ad hoc wireless networks: an information theoretic approach*, Foundations and Trends in Networking **1** (2006), no. 2, 145–270.