Clay Mathematics Proceedings
Volume **20**, 2014

Algebraic Approaches to FlipIt

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Abstract. This paper discusses a simple application of mathematics to a one person game which is, under various names, available on hand held game consoles, mobile phones, and similar devices. A linear algebra approach is presented in order to show properties of this puzzle. Using some more algebra, the question whether the special case where the game is played on a toric grid has a unique solution is briefly discussed.

Introduction

Various forms (in some cases merely describing special cases) of the game described in this paper exist under names such as Button Madness, Fiver, FlipIt, Lights Out, Magic Square, XL-25, Token Flip, and Orbix. The game can be described as follows. Take a finite, simple, undirected graph Γ. By definition, this is a finite nonempty set $v(\Gamma)$, and a finite set $e(\Gamma) \subset 2^{v(\Gamma)}$ (the power set of $v(\Gamma)$), such that every element of $e(\Gamma)$ has cardinality 2. The elements of $v(\Gamma)$ are called the vertices of Γ, and the elements of $e(\Gamma)$ are called the edges. For $a, b \in v(\Gamma)$, $b \in v(\Gamma)$ is called a neighbour of a if $\{a, b\} \in e(\Gamma)$.

The FlipIt game is now played as follows. The player is supposed to select a subset S of $v(\Gamma)$ in such a way, that for every $a \in v(\Gamma)$ we have

$$\#\left(S \cap (\{a\} \cup \{b \in v(\Gamma) \ : \ \{a, b\} \in e(\Gamma)\})\right) \equiv 1 \bmod 2.$$

An alternative way to describe this, begins by giving every vertex in the graph Γ the value 0 ("lights out"). The player consecutively selects vertices

$$v_1, v_2, \ldots, v_f$$

and selecting v_j results in switching the value of v_j and of each of its neighbours. So value 0 now becomes 1, and vice versa. Clearly, at the end of the sequence, a vertex v will have value 1 precisely when it happened an odd number of times that either it was selected, or one of its neighbours was selected. Now the aim of this game is to find a sequence $(v_j)_{1 \le j \le f}$ of vertices such that the value of each element of $v(\Gamma)$ is changed to 1.

The main result on the FlipIt game is that such a sequence exists:

2010 *Mathematics Subject Classification.* Primary 05C57, 15A06; Secondary 05C50, 91A43, 91A46.

Key words and phrases. Graphs, symmetric matrices, linear algebra, toric grid.

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THEOREM 0.1. *For every finite, simple, undirected graph* Γ*, the FlipIt game admits a solution.*

To our knowledge, no constructive proof of this result is known. In this note we present a non-constructive proof using only basic linear algebra over the field $\mathbb{F}_2$ consisting of precisely two elements. The properties of this field enter in an interesting way: over $\mathbb{F}_2$ – and only over $\mathbb{F}_2$ –, every homogeneous quadratic function is linear.

In a course at the Clay Mathematics Institute Summer School in Obergurgl (2012) by the first author of this paper, the main topic was a game-theoretic approach to the problem of resolution of singularities (see [**S**]). As an aside, several other games with a mathematical flavor were discussed, and FlipIt was one of them. The proof in this paper (which the first author learned from the webpage of the second author) was presented there.

The second author presented the same result in a lecture [**T**] aimed at (partly non-mathematical) friends and colleagues of the Groningen operator theorist Henk de Snoo, on the occasion of Henk's retirement (2010). With his son he published an expository note [**TT**] (also descibing constructive solutions for particular types of graphs) in the Dutch mathematical journal for high school students *Pythagoras*. Moreover he supervised two theses on the subject: a bachelor's thesis [**J**] dealing with uniqueness of solutions for particular types of graphs, and a master's thesis [**H**] investigating a technique from numerical linear algebra (notably the Block-Lanczos method) to find a solution. During the Clay Summer School mentioned above, an alternative, graph theoretic proof for the existence of solutions was presented by one of the participants.

We are very grateful to Norbert Pintye from Budapest, another participant of the summer school, who provided us with the following background on the FlipIt game. We only slightly supplemented his original email message with additional details.

> The game is due to a Hungarian research psychologist and popular science author László Mérő [**M1**], who not only invented the game in the early 1980s, but created it in reality, called XL-25 [**XL**]. It was presented at the International Game Expo, London, 1983. The first proof [**M2**] of the theorem - a purely graph theoretical one - is also due to him, and was published at the Mathematical and Physical Journal for Secondary Schools (KöMaL) in 1986.
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> The first proof using linear algebra is due to Klaus Sutner [**Su**]. The one presented below was also found (much earlier than we did) by the Eindhoven problem solvers group O.P. Lossers [**L**]; Theorem 0.1 above was actually proposed by Uri Peled from Chicago as a problem in the Amer. Math. Monthly in 1992. Including the one by the proposer, the Monthly received 42 solutions for it. The problem (and solutions to it) were known in Eindhoven before: in 1987, the Eindhoven mathematicians A. Blokhuis and H.A. Wilbrink proposed the linear algebra formulation in the problem session of the Dutch journal *Nieuw Archief voor Wiskunde*, see [**BW**]. The solution found by the proposers appeared 4 years later [**BW2**]; according to the journal, three others also solved the problem.

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A generalisation of the problem has been described in [**BR**].

The Eindhoven mathematician Andries Brouwer [**Br**] collected a lot of information on the problem, including various links and references.

1. Linear algebra

Let Γ be a finite, simple, undirected graph. We enumerate the vertices as

$$v(\Gamma) = \{1, 2, \dots, n\}.$$

It is clear from the description of the FlipIt game, that any permutation of the sequence $(v_j)_{1\le j\le f}$ of selected vertices results in the same set of vertices switching from 0 to 1. Moreover, if a vertex appears twice in the sequence, the result is the same as when two appearances of this particular vertex are removed from the sequence. In other words, the effect depends only on the unordered set

$$S := \{v \ : \ v \text{ appears an odd number of times in } (v_j)_{1\le j\le f}\}.$$

Such a set is described by an element $x = (\xi_1, \dots, \xi_n) \in \mathbb{F}_2^n$, with $\xi_j = 1 \Leftrightarrow j \in S$. The result of selecting precisely the vertices in S, or equivalently, the indices where x has a nonzero coordinate, is also described by an element $r \in \mathbb{F}_2^n$: its ith coordinate is nonzero precisely when vertex i has switched from 0 to 1 after selecting all vertices in the set S.

In this way, FlipIt on the graph Γ may be descibed as a map

$$\varphi \ : \ \{\text{subsets of } v(\Gamma)\} = \mathbb{F}_2^n \longrightarrow \{\text{possible results}\} \subseteq \mathbb{F}_2^n,$$

given by $\varphi(x) = r$. The question whether the game admits a solution translates into the existence of a vector $x \in \mathbb{F}_2^n$ such that $\varphi(x) = \mathbf{1}$, the vector with all coordinates equal to 1. Below we will prove Theorem 0.1, i.e., we claim that such a vector x exists.

LEMMA 1.1. *For a graph Γ as above, the associated map $\varphi : \mathbb{F}_2^n \to \mathbb{F}_2^n$ is linear over $\mathbb{F}_2$. With respect to the standard basis of $\mathbb{F}_2^n$, it is given by a symmetric matrix $A = (a_{i,j})$.*

PROOF. From the description of FlipIt, if $x, y \in \mathbb{F}_2^n$ correspond to sets of vertices $S_x = \{v_1, \dots, v_s\}$ and $S_y = \{w_1, \dots w_t\}$, respectively, then $x + y$ corresponds to $S_x \cup S_y \setminus (S_x \cap S_y)$. A vertex v is switched by selecting one of S_x, S_y and then the other, precisely when it is switched by only one of them. This shows $\varphi(x + y) = \varphi(x) + \varphi(y)$, implying that φ is linear over $\mathbb{F}_2$.

The ith column $(a_{1,i}, a_{2,i}, \dots, a_{n,i})^t$ of the matrix A shows the effect of only selecting the vertex i. The result of this is that i and all its neighbours in Γ switch from 0 to 1. Hence

$$a_{j,i} = 1 \Leftrightarrow i = j \text{ or } \{i, j\} \in e(\Gamma) \Leftrightarrow a_{i,j} = 1$$

which proves that A is symmetric. □

Note that the diagonal d of the matrix A has all its entries equal to 1. Hence Theorem 0.1 is a special case of the following more general result.

THEOREM 1.2. *Suppose $B = (b_{i,j})$ is a symmetric $n \times n$ matrix with coefficients in the finite field $\mathbb{F}_2$. Let $d = (d_{1,1}, d_{2,2}, \dots, d_{n,n})^t$ be the diagonal of B, which we regard as a column vector.*

Then the equation $Bx = d$ admits a solution $x \in \mathbb{F}_2^n$ (here x is a column vector).

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PROOF. One defines a bilinear form $\mathbb{F}_2^n \times \mathbb{F}_2^n \to \mathbb{F}_2$ by

$$(x, y) \mapsto x \cdot y := \sum \xi_j \eta_j$$

where $x = (\xi_1, \ldots, \xi_n)$ and $y = (\eta_1, \ldots, \eta_n)$. This form is nondegenerate, which means that if $x \cdot y = 0$ for every $y \in \mathbb{F}_2^n$, then x is the zero vector.

Given a nonempty $S \subset \mathbb{F}_2^n$, its orthogonal complement $S^\perp$ is defined by

$$S^\perp := \{x \in \mathbb{F}_2^n \ : \ x \cdot s = 0 \ \ \forall s \in S\}.$$

This is a linear subspace of $\mathbb{F}_2^n$. Since the bilinear form used here is nondegenerate, we have that $\dim S^\perp = n - \dim \operatorname{span}(S)$.

Moreover, for subspaces $V, W \subset \mathbb{F}_2^n$ it is true that

$$V \subset W \Leftrightarrow V^\perp \supset W^\perp$$

and

$$V^{\perp\perp} = V.$$

We have to prove that $Bx = d$ has a solution, in other words, that

$$\operatorname{span}(d) \subset B(\mathbb{F}_2^n).$$

This is equivalent to showing that

$$B(\mathbb{F}_2^n)^\perp \subset \{d\}^\perp.$$

To show that the latter inclusion indeed holds, let $y \in B(\mathbb{F}_2^n)^\perp$. Then in particular $y \cdot (By) = 0$. Writing $y = (\eta_1, \ldots, \eta_n)^t$, this means

$$0 = \sum_{i,j} b_{i,j} \eta_i \eta_j = \sum_{i=1}^{n} b_{i,i} \eta_i = d \cdot y,$$

since B is symmetric and we work over $\mathbb{F}_2$. This proves the result. □

REMARK 1.3. The argument given here shows (and uses) that the quadratic map given by $x \mapsto x \cdot (Bx)$ is linear over $\mathbb{F}_2$, and in fact it may also be described as $x \mapsto x \cdot d$. It is clear from this that for the proof it is crucial that we work over $\mathbb{F}_2$: over other fields, quadratic maps are not linear.

Evidently, the above existence proof is nonconstructive. Constructions of solutions, for example in case the graph Γ is an $m \times n$ rectangular grid with $m \leq 4$, can be found in [**TT**] and in [**DF+**].

EXERCISE 1.4. Provide a constructive solution on the $n \times n$ square grid.

2. Polynomial algebra

In a variant of the game, one may give any possible target $y \in \mathbb{F}_2^n$ and ask for x such that $\varphi(x) = y$. This game is solvable for every y if and only if the linear map $\varphi : \mathbb{F}_2^n \to \mathbb{F}_2^n$ is invertible, and this depends on the graph Γ. If φ is not invertible, then the probability of solvability is a (negative) power of 2, because the image of φ is an $\mathbb{F}_2$-vectorspace and therefore its cardinality is a power of 2.

We can compute these probabilities for certain graphs, called toric grids. For integers $u, v \geq 3$, the graph $T_{u,v}$ has vertex set $\{(i, j) \mid 0 \leq i < u, 0 \leq j < v\}$; there is an edge $\{(i_1, j_1), (i_2, j_2)\}$ if and only if ($i_1 = i_2$ and $j_1 - j_2 \equiv \pm 1 \mod v$) or ($j_1 = j_2$ and $i_1 - i_2 \equiv \pm 1 \mod u$). Each subset of $v(T_{u,v})$ corresponds to a unique polynomial in the polynomial algebra $R := \mathbb{F}_2[x, y]/\langle x^u - 1, y^v - 1\rangle$, namely the sum of all $x^i y^j$ with (i, j) in the subset. Stretching the language, we also denote

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by φ the linear function $R \to R$ that maps the polynomial $f = \sum_{(i,j)\in S} x^i y^j$ corresponding to a subset S, to the polynomial corresponding to $\varphi(S)$. Then φ is equal to multiplication by the polynomial $1 + x + x^{-1} + y + y^{-1} \in R$. Theorem 0.1 applied to the graph $T_{u,v}$ states that $t := \sum_{0\le i<u, 0\le j<v} x^i y^j$ is in the image of φ; in fact, in this special case we have $\varphi(t) = t$.

THEOREM 2.1. *The ratio of subsets of $v(T_{u,v})$ for which the above variant of FlipIt is solvable equals 2^{-r}, where r is the dimension of the quotient algebra $R/\langle 1 + x + x^{-1} + y + y^{-1}\rangle$. In particular, the variant is solvable for all subsets if and only if $r = 0$, i.e. $1 + x + x^{-1} + y + y^{-1}$ is invertible in R.*

PROOF. The image of φ is the principal ideal $I = \langle 1 + x + x^{-1} + y + y^{-1}\rangle$. Assume that its dimension as an $\mathbb{F}_2$-vectorspace is s. Then the ratio of elements in the image compared to elements in R is 2^{s-n}, where $n = uv$. On the other hand, $n - s$ is the dimension of the quotient algebra R/I. □

Here is a table showing the numbers r for $u, v \le 18$. We used the computer algebra [**M**] for the calculation. A more extensive table can be found on [**Br**].

u, v	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
3	4	4	2	6	2	4	4	4	2	6	2	4	4	4	2	6
4	4	0	0	8	0	0	4	0	0	8	0	0	4	0	0	8
5	2	0	8	2	0	0	2	8	0	2	0	0	10	0	0	2
6	6	8	2	8	2	8	6	4	2	12	2	4	6	8	2	8
7	2	0	0	2	0	0	14	0	0	2	0	0	2	0	0	14
8	4	0	0	8	0	0	4	0	0	16	0	0	4	0	0	8
9	4	4	2	6	14	4	4	4	2	6	2	16	4	4	2	6
10	4	0	8	4	0	0	4	16	0	4	0	0	12	0	0	4
11	2	0	0	2	0	0	2	0	0	2	0	0	2	0	0	2
12	6	8	2	12	2	16	6	4	2	16	2	4	6	16	2	12
13	2	0	0	2	0	0	2	0	0	2	0	0	2	0	0	2
14	4	0	0	4	0	0	16	0	0	4	0	0	4	0	0	28
15	4	4	10	6	2	4	4	12	2	6	2	4	12	4	18	6
16	4	0	0	8	0	0	4	0	0	16	0	0	4	0	0	8
17	2	0	0	2	0	0	2	0	0	2	0	0	18	0	16	2
18	6	8	2	8	14	8	6	4	2	12	2	28	6	8	2	8

A simpler special case of the game is obtained by using a chain consisting of $n \ge 3$ vertices. Each vertex i now corresponds to the monomial $x^i \in R_n := \mathbb{F}_2[x]/(x^n + 1)$. In the game, selecting the vertex i corresponds to adding $x^i(x + x^{-1} + 1)$ to the element already obtained. The fact that the game has a solution means that the element $t := \sum_{i=1}^{n} x^i$ is in the ideal $(x + x^{-1} + 1)R_n$. A different generator of this ideal is $x^2 + x + 1$. So the conclusion is that the game has a unique solution precisely when $\gcd(x^2 + x + 1, x^n + 1) = 1$ in the polynomial ring $\mathbb{F}_2[x]$, which is equivalent to $n \not\equiv 0 \bmod 3$. And in case $3|n$, the 'probability' of the variant of the game described above equals $1/4$: indeed, $\#R_n/(x^2 + x + 1) = \#\mathbb{F}_2[x]/(x^2 + x + 1) = 4$.

As the above table suggests, the situation is not as simple for the toric grids $T_{u,v}$. We will now briefly indicate how one can describe in terms of the integers u, v, the cardinality of the set of solutions to the FlipIt game.

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The polynomial algebra $R = R_{u,v} := \mathbb{F}_2[x,y]/(x^u+1, y^v+1)$ is a finite, commutative ring, hence in particular every element in $R_{u,v}$ is either a unit or a zero-divisor. For $a, b \geq 0$, the kernel of the reduction homomorphism

$$\pi \ : \ R_{2^a u, 2^b v} \longrightarrow R_{u,v}$$

consists of nilpotent elements, hence an element $\alpha \in R_{2^a u, 2^b v}$ is a unit if and only if its image $\pi(\alpha) \in R_{u,v}$ is a unit. Applying this to $1 + x + x^{-1} + y + y^{-1}$, the conclusion is that the FlipIt game has a unique solution for $T_{u,v}$ if and only if it has a unique solution for $T_{2^a u, 2^b v}$.

This reduces the problem of unique solvability for toric grids to the case where both u and v are odd. In this case, $R_{u,v}$ is the coordinate ring of the finite set $\mu_u \times \mu_v$ consisting of all pairs $(\alpha, \beta) \in \overline{\mathbb{F}_2} \times \overline{\mathbb{F}_2}$ satisfying $\alpha^u = \beta^v = 1$ (Here $\overline{\mathbb{F}_2}$ denotes a fixed algebraic closure of $\mathbb{F}_2$). Hence $f \in R_{u,v}$ is a unit precisely when $f(\alpha, \beta) \neq 0$ for all $(\alpha, \beta) \in \mu_u \times \mu_v$. Applying this to

$$g := 1 + x + x^{-1} + y + y^{-1} \in R_{u,v},$$

a direct consequence is: suppose $3|u$. Take $(\omega, 1) \in \mu_u \times \mu_v$ where ω is a primitive cube root of 1. Then $g(\omega, 1) = 0$, hence the Flipit game on $T_{u,v}$ has more than one solution whenever $3|uv$.

Similarly, suppose $5|u$ and $5|v$. Taking a primitive 5th root of unity ζ, one has $g(\zeta, \zeta^2) = 0$ hence also in this case, the game has more than one solution.

A general statement in the same spirit is: if the FlipIt game on $T_{u,v}$ has more than one solution, then it also has several solutions on $T_{u',v'}$ whenever $u|u'$ and $v|v'$. This is true since a zero of g in $\mu_u \times \mu_v$ is obviously also a zero of g in the larger set $\mu_{u'} \times \mu_{v'}$.

To say slightly more, suppose $\zeta_n, \zeta_m \in \overline{\mathbb{F}_2}$ have exact order n and m respectively (so they are primitive nth and mth roots of unity) and suppose $g(\zeta_n, \zeta_m) = 0$. This implies in particular that the degree of the field $\mathbb{F}_2(\zeta_n, \zeta_m)$ over each of the subfields $\mathbb{F}_2(\zeta_m)$ and $\mathbb{F}_2(\zeta_n)$ is at most 2. As a consequence,

$$\frac{[\mathbb{F}_2(\zeta_m) : \mathbb{F}_2]}{[\mathbb{F}_2(\zeta_n) : \mathbb{F}_2]} \in \{\frac{1}{2}, 1, 2\}.$$

Since the extension field of $\mathbb{F}_2$ generated by a primitive kth root of unity has degree $\mathrm{ord}(2,k) :=$ the order of 2 mod k in the group $(\mathbb{Z}/k\mathbb{Z})^\times$, one concludes

$$\frac{\mathrm{ord}(2,m)}{\mathrm{ord}(2,n)} \in \{\frac{1}{2}, 1, 2\}.$$

As an example, the toric grid $T_{5,7}$ admits a unique solution. Namely, $g(1, \alpha)$ and $g(\beta, 1)$ are zero only for α, β of order 3, which do not occur in μ_5 or μ_7. And $g(\zeta_5, \zeta_7) \neq 0$ since $\mathrm{ord}(2,5)/\mathrm{ord}(2,7) = 4/3$.

As a final remark, for u, v odd, the dimension r of the quotient algebra R/gR introduced above, equals the cardinality of the set

$$V(g) := \{(\alpha, \beta) \in \mu_u \times \mu_v \ : \ g(\alpha, \beta) = 0\}.$$

So the toric grid cases of FlipIt allows a nice interpretation which may be described in terms of elementary algebraic geometry.

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