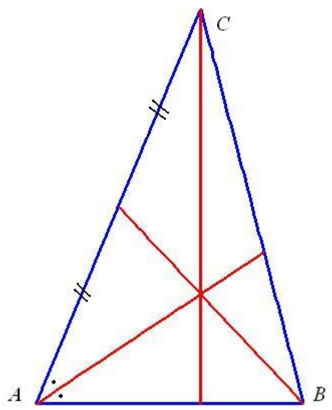
TRIANGLES AND A CUBIC CURVE

JAAP TOP

In this note another application of arithmetic of cubic curves to a problem involving triangles is discussed. The best known such application deals with congruent numbers. The present application is of a much simpler nature.

Recall that in a triangle ABC, a *bisector* (or, angle bisector) is a line through one of the vertices which divides the corresponding angle into equal parts. Similarly, a *median* is a line through one of the vertices which divides the opposite side into equal parts. And finally, an *altitude* is a line through one of the vertices which is perpendicular to the opposite side. In college geometry one learns some facts concerning these lines in a triangle. For instance, the three medians of a triangle are concurrent, the three altitudes are concurrrent, and so are the three (interior) bisectors.

Th.J. Kletter & F. van der Blij (Gorssel, the Netherlands) asked for triangles ABC with the property that a bisector in A, the median in B and the altitude in C are concurrent. It is easy to construct such a triangle starting from an arbitrary angle CAB and an arbitrary length $|AC|$: the altitude in C and the bisector in A are determined from this data, hence also two points of the remaining median, namely the midpoint of AC and the point of the given altitude and bisector. So we know the median, and hence the point B.



To make the problem more interesting (and to be able to present concrete examples suitable for high school students), Kletter & Van der Blij added the following restriction:

Describe the triangles as above, which moreover have the property that the lengths of the three sides are rational numbers.

Note that give any such triangle, scaling it by a rational factor yields another one. So we may as well ask for all examples upto scaling. Also, note that at least one example exists, namely the equilateral triangle.

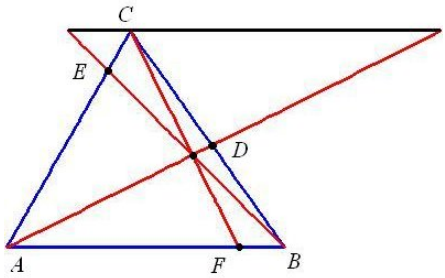
To put the problem into a more convenient form, two classical results from elementary geometry will be used. The first one is due to Giovanni Ceva, who proved it in 1678:

CEVA'S THEOREM. *Given a triangle ABC with a point D on side BC, a point E on side AC and a point F on side AB.*

Then the lines AD, BE and CF are concurrent, if and only if

$$\frac{|AF|}{|FB|} \cdot \frac{|BD|}{|DC|} \cdot \frac{|CE|}{|EA|} = 1.$$

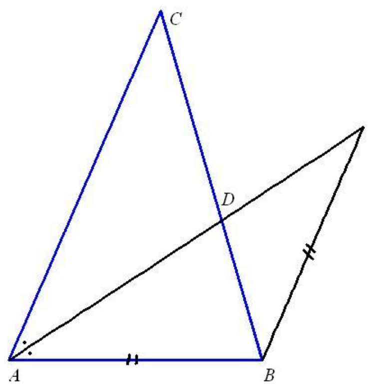
To show this, use a picture as presented below. Surprisingly, although many results on triangles date back to Euclid's Elements, this one was discovered in the late 17th century!



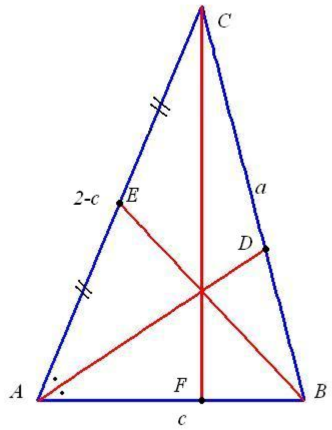
The second classical result is the following. The picture below suggests how it is proven.

PROPOSITION 3 IN EUCLID, ELEMENTS, VI. *Suppose that in triangle ABC the point D is on the side BC. Then AD is a bisector, if and only if*

$$\frac{|AB|}{|AC|} = \frac{|BD|}{|CD|}.$$



Now we proceed in describing all possible such triangles. Put $a = |BC|, b = |AC|$ and $c = |AB|$. Without loss of generality, we can assume $b + c = 2$.



The two results explained above now show

$$\frac{|AF|}{c - |AF|} = \frac{2-c}{c},$$

which implies $|AF| = c(2-c)/2$ and $|BF| = c^2/2$.

Using Pythagoras's theorem in the triangles AFC and BFC gives

$$a^2 = c^3 - 4c + 4.$$

Conversely, any pair of rational numbers (a, c) satisfying $0 < c < 2$ and $a^2 = c^3 - 4c + 4$ gives a triangle as desired, namely with sides of length $|a|$, $b = 2 - c$ and c, respectively (the fact that these numbers satisfy the triangle inequality follows using the equality $a^2 = c^3 - 4c + 4$). Note that for different c's, we obtain nonsimilar triangles. So the initial problem is reduced to the problem:

find all rational points (c, a) on the cubic curve C with equation $y^2 = x^3 - 4x + 4$, which have the additional property $0 < c < 2$.

Note that $P := (2, 2) \in C(\mathbb{Q})$. In the group $C(\mathbb{Q})$ we compute some multiples nP using the public source program GP-Pari:

```
? for(n=1,12,print(ellpow(e,[2,2],n)))
[2, 2]
[0, 2]
[-2, -2]
[1, -1]
[6, -14]
[8, 22]
[10/9, 26/27]
[-7/4, 19/8]
[-6/25, -278/125]
```

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[88/49, -554/343]
[310, -5458]
[273/121, 3383/1331]
?
```

The point $4P = (1, -1)$ corresponds to the triangle with sides $(1, 1, 1)$. This is the equilateral triangle, which we already knew as an example.

The point $7P = (10/9, 26/27)$ corresponds, after rescaling, to a triangle with sides $(13, 12, 15)$. So we have a first nontrivial example!

Another one is given by $10P$. The corresponding triangle has sides $(308, 35, 277)$.

Of course we could continue in this way and possibly find some more examples. Instead, we will apply some of the theory of cubic curves. Firstly, since $7P$ has coordinates which are not integers, we know by the Nagell-Lutz Theorem that $P \in C(\mathbb{Q})$ has infinite order. This leads to the question, how many of the points nP satisfy $0 < x(nP) < 2$.

To answer the latter question, we will consider the real points $C(\mathbb{R})$. These points also form a group, with $C(\mathbb{Q})$ (and hence $\mathbb{Z} \cdot P$) as a subgroup. It turns out that $C(\mathbb{R})$ is a very easy group:

Put

$$\mathbb{T} := \{z \in \mathbb{C}^* \; ; \; |z| = 1\} \subset \mathbb{C}^* \text{ (the circle group).}$$

Take $\alpha \in \mathbb{R}$ the unique real zero of $x^3 - 4x + 4$, so

$$\alpha \approx -2,3829757679062374941227085 36 \cdots .$$

Finally, put

$$\Omega := 2 \int_\alpha^\infty \frac{dx}{\sqrt{x^3 - 4x + 4}}.$$

With these notation, define a map

$$\Psi := e^{2\pi i \varphi} \; : \; E(\mathbb{R}) \longrightarrow \mathbb{T}$$

by $\Psi(O) = 1$ and

$$\varphi : \; (x, y) \mapsto \varphi(x, y) := \begin{cases} \frac{1}{\Omega} \int_x^\infty \frac{dt}{\sqrt{t^3 - 4t + 4}} & \text{for } y \geq 0; \\ 1 - \frac{1}{\Omega} \int_x^\infty \frac{dt}{\sqrt{t^3 - 4t + 4}} & \text{for } y \leq 0. \end{cases}$$

From these definitions, it is clear that if $\alpha \leq x_1 < x_2$, then for $y_i := \sqrt{x_i^3 - 4x_i + 4}$ we have

$$\frac{1}{2} \geq \varphi(x_1, y_1) > \varphi(x_2, y_2) > 0$$

and

$$\frac{1}{2} \leq \varphi(x_1, -y_1) < \varphi(x_2, -y_2) < 1.$$

It follows that Ψ is injective. Moreover, since φ is continuous and

$$\lim_{x\to\infty} \varphi(x, \sqrt{x^3 - 4x + 4}) = 0,$$

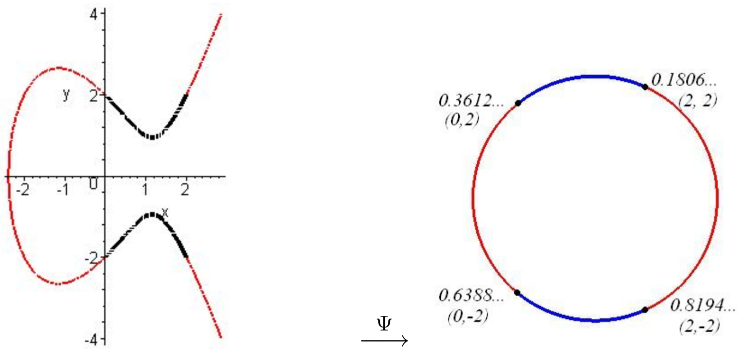
we conclude that Ψ is also surjective, hence Ψ is invertible.

A stronger fact about Ψ is that it is an isomorphism between the groups $C(\mathbb{R})$ and $\mathbb{T}$. To prove this, we only have to show that if $P_1 + P_2 + P_3 = O$, then $\varphi(P_1) + \varphi(P_2) + \varphi(P_3) \in \mathbb{Z}$. This follows from a straightforward calculation with differentials: if $(x, y) + Q = (x', y')$, then

$$\frac{dx}{y} = \frac{dx'}{y'}.$$

We will omit the calculation.

The consequence of this is, that $\Psi(P)$ generates an infinite, cyclic subgroup of $\mathbb{T}$. The classical equidistribution theorem of Hermann Weyl (1909) then implies that this subgroup is dense in $\mathbb{T}$. It follows that $\mathbb{Z} \cdot P$ is dense in $C(\mathbb{R})$. In particular, for infinitely many integers n we have that $0 < x(nP) < 2$. So there exist infinitely many pairwise nonsimilar triangles with the desired property.



In fact, Weyl's equidistribution theorem states that the proportion of the set of all n such that $\Psi(P)^n$ is in a certain interval, equals the ratio of the length of that interval over the length of the circle. In our case this implies that a little more than 36% of all integers n satisfy $0 < x(nP) < 2$.