Jean-Pierre Serre:

the first Abel prize recipient (2003).

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The Abel Prize 2003

The Norwegian Academy of Science and Letters has decided to award the Abel Prize for 2003 to:

Jean-Pierre Serre

Collège de France, Paris, France,

"for playing a key role in shaping the modern form of many parts of mathematics, including topology, algebraic geometry and number theory".

Prize amount: NOK 6.000.000 (760.000 Euro)

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born 15 September 1926

PhD in 1951 ("*Homologie singulière des espaces fibrés. Applications*")

supervisor: Henri Cartan (Sorbonne, Paris)

1956–1994 professor in Algebra & Geometry at the Collège de France (Paris)

Collected papers (4 volumes) contain 173 items (including many letters and abstracts of courses given at the Collège de France)

13 books

Most recent text in Collected papers: 1998.

Most recent text according to MathSciNet: 2006.

many distinctions

honorary degrees from Cambridge, Stockholm, Glasgow, Athens, Harvard, Durham, London, Oslo, Oxford, Bucharest, Barcelona

honorary member or foreign member of many Academies of Science (including KNAW, 1978)

Many Prizes (Fields Medal, Prix Gaston Julia, Steele Prize, Wolf Prize, $\cdots\cdots$)

1954, ICM Amsterdam

Hermann Weyl presented the Fields Medals to Kunihiko Kodaira (1915-1997) and to Serre commentary by Serre (email of 27 December 2004):

"$\cdots\cdots$ *I barely recognize myself on the picture where papy Hermann Weyl seems to tell me (and Kodaira): "Naughty youngsters! It is OK this time, but don't do it again ! " And he gave my medal to Kodaira, and Kodaira's medal to me, so that we had to exchange them the next day.*"

Fields medal, 3 years after his PhD thesis, for two reasons:

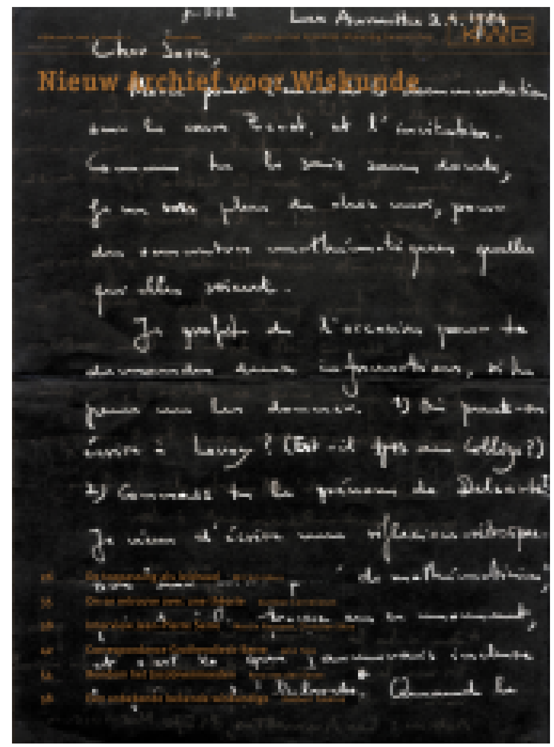
1) The thesis work (introducing ‘spectral sequences’ in algebraic topology; in particular the ‘Serre spectral sequence’);

2) Introducing ‘sheaf theory’ in complex analytic geometry.

(commercial break)

Serre is an interested reader of the 5th Series of Nieuw Archief:

the quote above is part of his reaction to the 1954 ICM pictures published in NAW in December 2004;
he sent us several original letters from him to Alexander Grothendieck, and from Grothendieck to him, to be used with a text John Tate wrote for NAW on the Grothendieck-Serre correspondence (March 2004)



Serre and sports:

apart from skiing and rock climbing, used to be a quite good table tennis player (but needed an excuse, age difference, when finally losing from, e.g., Toshiyuki Katsura, 1989, Texel).

experimental ‘science’

versus

three conjectures

Serre: "*Une conjecture est d'autant plus utile qu'elle est plus précise, et de ce fait testable sur des exemples.*"

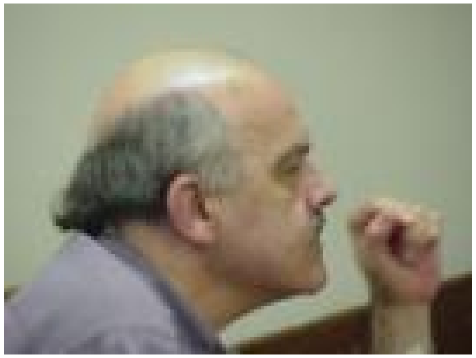
Serre's problem on projective modules

1955, problem stated in the paper *Faisceaux Algébriques Cohérents*:

is every projective module M *over a polynomial ring* $R = K[x_1, \ldots, x_n]$ *(with* K *a field), free?*

(projective means that M is a direct summand of a free module: $M \oplus N \cong R^n$ for some module N and integer n)

Answered independently by D. Quillen and A. A. Suslin (1976): YES!



Remark: over many other rings, projective $\neq$ free!

Example 1: $R := \mathbb{Z}[\sqrt{-5}]$, $M := \{a + b\sqrt{-5} \in R\ ;\ a \equiv b \bmod 2\}$.

then M is not free;

and $R^2 \cong M \oplus M$ via the map

$$(f, g) \mapsto (2f + (1+\sqrt{-5})g, (1-\sqrt{-5})f + 2g)$$

(so M is projective)

Example 2, more geometric (Möbius strip):

$R := \{f \in C^\infty(\mathbb{R})\ ;\ f(x+2\pi) = f(x)\}$

(the ring of real C^∞-functions on the circle)

$M := \{m \in C^\infty(\mathbb{R})\ ;\ m(x+2\pi) = -m(x)\}$

As in the previous example, M is not free, but $R^2 \cong M \oplus M$,

via $(f(x), g(x)) \mapsto$
$(f(x)\cos(x/2) + g(x)\sin(x/2), f(x)\sin(x/2) - g(x)\cos(x/2))$.

Serre's conjecture on modular forms (1987).

$p(x) \in \mathbb{Z}[x]$ monic, irreducible,

over $\mathbb{C}$: $p(x) = \prod(x - \alpha_j)$;

$K := \mathbb{Q}(\alpha_1, \ldots)$ field extension generated by the zeroes of $p(x)$;

$\mathrm{Gal}(K/\mathbb{Q})$: the (finite) group of field automorphisms of K;

$\rho : \mathrm{Gal}(K/\mathbb{Q}) \hookrightarrow \mathrm{GL}_2(\mathbb{F}_q)$ embedding into group of invertible 2×2 matrices over some finite field, with assumptions:

1. Take $c \in \mathrm{Gal}(K/\mathbb{Q})$ complex conjugation restricted to K. Then $\det(\rho(c)) = -1 \in \mathbb{F}_q$;

2. ρ is irreducible, i.e., there is no 1-dimensional linear subspace $V \subset \mathbb{F}_q^2$ such that $\rho(g)$ sends V to V for every $g \in \mathrm{Gal}(K/\mathbb{Q})$.

Conjecture (Serre): this situation arises from a modular form.

The work towards understanding this conjecture has been fundamental in, e.g., Wiles' proof of Fermat's Last Theorem

(work of Ribet, Edixhoven, quite recently Khare, Wintenberger, Dieulefait) modular form: certain analytic function

$$\mathbb{H} := \{z \in \mathbb{C} \, ; \, \mathrm{im}(z) > 0\} \to \mathbb{C},$$

given by Fourier expansion $f(z) = q + a_2 q^2 + \ldots$, with $q = e^{2\pi i z}$, $z \in \mathbb{H}$,

$f(\frac{az+b}{cz+d}) = \epsilon(d)(cz+d)^k f(z)$ for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ with c a multiple of N

The integer $N > 0$ is called the level of f,
the integer $k > 0$ is called the weight of f

$\epsilon : (\mathbb{Z}/N\mathbb{Z})^* \to \mathbb{C}^*$ is called the character of f

$\rho \;:\; \mathsf{Gal}(K/\mathbb{Q}) \hookrightarrow \mathsf{GL}_2(\mathbb{F}_q)$ 'arises from the modular form f' means (somewhat imprecise):

there exists $\varphi : \mathbb{Z}[a_2, a_3, a_4, \ldots] \to \mathbb{F}_q$ such that

$$\mathrm{trace}(\rho(\mathrm{Fr}_\ell)) = \varphi(a_\ell)$$

for all but finitely many prime numbers ℓ.

To define Fr_ℓ: take splitting field $\mathbb{F}_{\ell^n}$ of $p(x)$ mod ℓ;
construct $\mathbb{Z}[\alpha_1, \alpha_2, \ldots] \to \mathbb{F}_{\ell^n}$;
'lift' the field automorphism $\xi \mapsto \xi^\ell$ of $\mathbb{F}_{\ell^n}$ to an automorphism Fr_ℓ of $\mathbb{Z}[\alpha_1, \alpha_2, \ldots]$ and of the field K.

Serre gives a recipe that, given K and ρ, defines a 'minimal' level N (with $\gcd(N, q) = 1$), a minimal weight k, and the character ϵ.

In the 90's Ribet, Mazur, Carayol, Diamond, Edixhoven and others proved, that if K and ρ arise from some modular form (with level coprime to q), then also from one with the level and weight predicted by Serre.

A very simple example:

K is the extension (degree 6) of $\mathbb{Q}$ generated by the roots of $x^3 - 4x + 4 = 0$.

Then $K = \mathbb{Q}(\alpha, \sqrt{-11})$ with α any of the three roots;

$\mathrm{Gal}(K/\mathbb{Q}) \cong S_3$ (all permutations of the three roots);

Take any isomorphism $\rho : \mathrm{Gal}(K/\mathbb{Q}) \to \mathrm{GL}_2(\mathbb{F}_2)$

The pair (K, ρ) arises from the modular form

$$f(z) = q \prod_{n=1}^{\infty} (1 - q^n)^2 (1 - q^{11n})^2.$$

this means: write $f(z) = \sum_{m=1}^{\infty} a_m q^m =$

$q - 2q^2 - q^3 + 2q^4 + q^5 + 2q^6 - 2q^7 - 2q^9 - 2q^{10} + q^{11} - 2q^{12} + 4q^{13} + \ldots$

For $\ell \neq 2, \neq 11$ a prime number:

$$\begin{gathered} a_\ell \text{ is odd} \\ \Leftrightarrow \\ x^3 - 4x + 4 \text{ is irreducible } \mod \ell \\ \Rightarrow \\ \ell \equiv 1, 3, 4, 5, 9 \bmod 11 \end{gathered}$$

Also for the primes $\ell \neq 2, \neq 11$:

$$x^3 - 4x + 4 \bmod \ell \text{ splits in three linear factors}$$
$$\Leftrightarrow$$
$$a_\ell \text{ is even } \& \ \ell \equiv 1, 3, 4, 5, 9 \bmod 11$$

For odd primes $\ell \equiv 2, 6, 7, 8, 10 \bmod 11$, the number a_ℓ is even and $x^3 - 4x + 4 \bmod \ell$ has a linear and a quadratic irreducible factor.

Serre's conjecture on rational points on curves of genus 3 over a finite field

(1985, course given at Harvard)

Finite field $\mathbb{F}_q$ (for simplicity: q odd).

Curve C of genus 3 over $\mathbb{F}_q$:

either hyperelliptic, which means a complete curve (so, including two points 'at infinity'), given by an equation $y^2 = f(x)$, with $f(x)$ a polynomial of degree 8 over $\mathbb{F}_q$ without multiple factors;

or a nonsingular curve in $\mathbb{P}^2$ given by a quartic equation over $\mathbb{F}_q$.

more generally, a nonsingular curve in $\mathbb{P}^2$ given by an equation of degree d has genus $g = (d-1)(d-2)/2$.

The set of points on C with coordinates in $\mathbb{F}_q$ is denoted $C(\mathbb{F}_q)$.

H. Hasse and A. Weil: $\#C(\mathbb{F}_q) \leq q + 1 + 2g\sqrt{q}$ (here g is the genus of C)

Improvement by Serre (1985): $\#C(\mathbb{F}_q) \leq q + 1 + g[2\sqrt{q}]$ (here $[x]$ denotes the largest integer $\leq x$)

Serre's conjecture: these bounds are sharp for $g = 3$, in the following sense: there should exist an integer ϵ such that, for every finite field $\mathbb{F}_q$, a curve C of genus $g = 3$ over $\mathbb{F}_q$ exists with

$$\#C(\mathbb{F}_q) \geq q + 1 + g[2\sqrt{q}] - \epsilon$$

The analogous statement for $g = 1$ is true (M. Deuring, 1940's)

Serre proves the analogous statement for $g = 2$ in his Harvard course (1985)

A similar conjecture for $g >> 0$ cannot be expected: fix q and compare

$$\lim_{g\to\infty} \frac{q + 1 + g[2\sqrt{q}] - \epsilon}{g} = [2\sqrt{q}]$$

with a theorem of Drinfeld & Vladut (1983):

$$\limsup_{g\to\infty} \frac{\max_{C \text{ of genus } g} \#C(\mathbb{F}_q)}{g} \leq \sqrt{q} - 1$$

some results towards this conjecture:

1) Ibukiyama, 1993: restrict to $\mathbb{F}_q$ with q odd, q a square but not a fourth power. Then the conjecture holds, with $\epsilon = 0$.

2) with R. Auer, 2002: restrict to $\mathbb{F}_q$ with $q = 3^n$, all $n \geq 1$. Then the conjecture holds, with $\epsilon = 21$.

3) Serre & Lauter, 2002 and independently Auer & Top, 2002: take $\epsilon = 3$ (we: $\epsilon = 21$). For every finite field $\mathbb{F}_q$, there exists a curve C over $\mathbb{F}_q$ of genus 3 for which either

$$\#C(\mathbb{F}_q) \geq q + 1 + g[2\sqrt{q}] - \epsilon$$

or

$$\#C(\mathbb{F}_q) \leq q + 1 - g[2\sqrt{q}] + \epsilon$$

(but cannot decide which of the two...)