A tutorial on the informativity framework for data-driven control

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Abstract—The purpose of this paper is to provide a tutorial on the so-called informativity framework for direct data-driven control. This framework views data-driven analysis and design through the lens of robust control, and aims at assessing system properties and determining controllers for sets of systems unfalsified by the data. We will first introduce the informativity approach at an abstract level. Thereafter, we will study several case studies where we highlight the strength of the approach in the context of stabilizability analysis and stabilizing feedback design in different setups involving exact and noisy data, and for both input-state and input-output measurements. Finally, we provide an account of other applications of the data informativity framework.

I. INTRODUCTION

Roughly speaking, systems and control theory deals with the problem of making a concrete physical system behave according to certain desired specifications. In order to achieve this desired behavior, the system can be interconnected with a physical device, called a controller. The problem of finding a mathematical description of such a controller is called the control design problem.

In order to obtain a mathematical description of a controller for a to-be-controlled physical system, a possible first step is to obtain a mathematical model of the physical system. Such a mathematical model can take many forms. For example, the model could be in terms of ordinary or partial differential equations, difference equations, or transfer matrices.

There are several ways to obtain a mathematical model for the physical system. The usual way to get a model is to apply the basic physical laws that are satisfied by the variables appearing in the system. This method is called first principles modeling. For example, for electro-mechanical systems, the set of basic physical laws that govern the behavior of the variables in the system (conservation laws, Newton’s laws, Kirchhoff’s laws, etc.) form a mathematical model.

An alternative way to obtain a model is to do experiments on the physical system: certain external variables in the physical system are set to take particular values, while at the same time other variables are measured. In this way, one obtains data on the system that can be used to find mathematical descriptions of laws that are obeyed by the system variables, thus obtaining a model. This method is called system identification.

The second step in a control system design problem is to decide which desired behavior we would like the physical system to have. Very often, this desired behavior can be formalized by requiring the mathematical model to have certain qualitative or quantitative mathematical properties. Together, these properties form the design objective.

Based on the mathematical model of the physical system and the design objective, the third, ultimate, step is to design a mathematical model of a suitable controller. This approach, leading from a model and a design objective (or list of design specifications) to a model of a controller is an important paradigm in systems and control, and is often called model-based control. Indeed, many existing control design techniques rely on a system model, represented by, for example, a state-space system or transfer matrices.

In this paper, we will deal with an approach to control design that circumvents the step of finding a mathematical model of the to-be-controlled physical system. This alternative approach deals with the problem of designing control laws directly on the basis of measured data, and is called the data driven approach to control design. Of course, one can argue that also the combination of system identification followed by model based control as described above is an instance of data driven control design. Indeed, methods using this combination are often called indirect methods of data-driven control, consisting of the two-step process of data-driven modeling (i.e., system identification) followed by model-based control.

In addition to the above indirect methods, we distinguish direct methods to data driven control design [1]–[3]. These direct approaches focus on directly mapping data to controllers without an intermediate step of system identification. Both paradigms have different pros and cons. For example, identification might be expensive and the obtained model may not always be useful for the intended control design problem [4]. In addition, in many situations unique system identification might be difficult or even impossible because the data are corrupted by noise, or in other ways simply do not give sufficient information. In contrast, direct data-driven control design has the premise of being an end-to-end approach, requiring less expert knowledge, and thus often being the preferred choice for practitioners. However, in comparison to the maturity of system identification, the theory of direct data-driven control is still in its infancy.

Inspired by the latter observation, this paper is dedicated to developing theory within the direct approach to data-driven control design [3]. The general problem that will be considered is to use only the data obtained from the unknown physical system to verify its system theoretic properties and
to construct controllers.

As noted above, in some situations, the data obtained from the physical system contain sufficient information to identify the system model uniquely. It is not surprising that in such situations control design can be based on the data directly. In general however, the data will not contain sufficient information to uniquely identify the system model. One can then argue that the current state of the art in system identification not only provides methods to identify a system model, but also aims at obtaining descriptions of the uncertainty about this true model. Unfortunately, these uncertainty descriptions are not always compatible with those that can be handled by existing robust control techniques [4].

Intriguing questions are therefore the following: is it possible to verify system properties and/or to obtain controllers from data that do not contain sufficient information to uniquely identify the system? And, in addition, is it possible to do control design in situations that the uncertainty descriptions obtained from system identification are not compatible with currently existing robust control methods?

The answer to the above questions depends on the interplay between the given data and the particular system property or control design problem at hand. In many situations, the answer will turn out to be affirmative. If, for a certain system property or control design problem, the data contain sufficient information to verify that property or to design a controller, then we will call the data informative for this system property or control design problem. In such situations direct data driven analysis and control is a powerful alternative for the combination of system identification and model-based analysis and control.

The concept of data informativity will play the central role in this paper. It will be shown to constitute a powerful framework that can be applied to a large number of system analysis and control design problems.

II. DATA INFORMATIVITY FRAMEWORK

In this section we will introduce the concept of data informativity for verifying a given system property or solving a certain control design problem at a fairly abstract level.

To start with, we fix a certain model class $\mathcal{M}$. This model class is a given set of systems that is assumed to contain the ‘true’ system (i.e., a mathematical model of the underlying unknown physical system), denoted by $S$. We assume that the true system $S$ is not known but that we do have access to a set of data, $\mathcal{D}$, which is generated by this system. As explained in the introduction, we are interested in assessing system-theoretic properties of $S$ and designing control laws for it from the data $\mathcal{D}$. Given the set of data $\mathcal{D}$, we define $\Sigma_\mathcal{D} \subseteq \mathcal{M}$ to be the set of all systems that are consistent with the data $\mathcal{D}$, i.e., that could also have generated the same data. In other words, it is impossible to distinguish the true system $S$ from any other system in $\Sigma_\mathcal{D}$ on the basis of the given data $\mathcal{D}$ alone.

We will now first focus on data-driven analysis of system theoretic properties. Let $P$ be some system theoretic property. We will denote the set of all systems within $\mathcal{M}$ having this property by $\Sigma_p$. Suppose we are interested in the question whether our true system $S$ has the property $P$. Since the only information we have to base our answer on are the data $\mathcal{D}$ obtained from the true system, we can only conclude from the data that the true system has property $P$ if all systems consistent with the data $\mathcal{D}$ have the property $P$. If this is the case, we call the data informative for the system property. This leads to the following definition.

**Definition 1 (Informativity for analysis):** We say that the data $\mathcal{D}$ are informative for property $P$ if $\Sigma_\mathcal{D} \subseteq \Sigma_P$, i.e., all systems that are compatible with the data have property $P$.

In general, if the true system $S$ can be uniquely determined from the data $\mathcal{D}$, that is $\Sigma_\mathcal{D} = \{S\}$ and $S$ has the property $P$, then it is evident that the data $\mathcal{D}$ are informative for $P$. However, the converse may not be true: $\Sigma_\mathcal{D}$ might contain many systems, all of which have property $P$. In the data informativity framework, we are interested in necessary and sufficient conditions for informativity of the data. Such conditions reveal the minimal amount of information required to assess the property $P$. A natural problem statement is therefore the following:

**Problem 1 (Informativity problem for analysis):** Provide necessary and sufficient conditions on the data $\mathcal{D}$ under which these data are informative for property $P$.

The above gives us a general framework to deal with data-driven analysis problems. We will also deal with data-driven control problems. The objective in such problems is the data-based design of controllers such that the closed loop system, obtained from the interconnection of the true system $S$ and the controller, satisfies the given control objective. As for the analysis problem, we have only the information from the data to base our design on. Therefore, we can only guarantee that our control objective is achieved if the designed controller achieves the design objective when interconnected with any system from the set $\Sigma_\mathcal{D}$.

For the framework to allow for data-driven control problems, we will consider a given control objective $O$ (for example, a system theoretic property or a guaranteed performance of the closed loop system). Denote by $\Sigma_O$ the set of all systems that satisfy the control objective $O$. For a given controller $K$, denote by $\Sigma_\mathcal{D}(K)$ the set of all systems
obtained as the interconnection of a system in $\Sigma_D$ with the controller $K$. We then have the following variant of informativity:

**Definition 2 (Informativity for control):** We say that the data $D$ are informative for the control objective $O$ if there exists a controller $K$ such that $\Sigma_D(K) \subseteq \Sigma_O$.

Obviously, the first step in any data-driven control problem is to determine whether it is possible to obtain, from the given data, a suitable controller. This leads to the following informativity problem:

**Problem 2 (Informativity problem for control):** Provide necessary and sufficient conditions on $D$ under which the data are informative for the control objective $O$.

The second step of data-driven control involves the design of a suitable controller. In terms of our framework, this can be stated as:

**Problem 3 (Control design problem):** Under the assumption that the data $D$ are informative for the control objective $O$, find a controller $K$ such that $\Sigma_D(K) \subseteq \Sigma_O$.

In the next section, we highlight the strength of the informativity framework by applying it to a selection of analysis and design problems.

### III. Case Studies

We will now illustrate how the data informativity framework can be employed to deal with stabilizability analysis and stabilizing feedback design based on data.

#### A. Stabilizability analysis from exact (noise free) data

For given $n$ and $m$, let the model class $M$ be the set of all discrete-time linear input/state systems of the form

$$x(t + 1) = Ax(t) + Bu(t),$$

where $x$ is the $n$-dimensional state and $u$ is the $m$-dimensional input. Let the true system $S$ be represented by the matrices $(A_s, B_s)$.

An example of a data set $D$ arises when considering data-driven problems on the basis of input and state measurements. Suppose that we collect input-state data on the time interval $[0, T]$. Let

$$U_- := [u(0) \ u(1) \ \cdots \ u(T - 1)],$$

$$X := [x(0) \ x(1) \ \cdots \ x(T)],$$

denote the samples on this time interval. By defining

$$X_- := [x(0) \ x(1) \ \cdots \ x(T - 1)],$$

$$X_+ := [x(1) \ x(2) \ \cdots \ x(T)],$$

we clearly have $X_+ = A_sX_- + B_sU_-$ because the true system is assumed to generate the data.

We then define the data as $D := (U_-, X)$. In this case, the set $\Sigma_D$ is equal to $\Sigma_{(U_-, X)}$ defined by

$$\Sigma_{(U_-, X)} := \{(A, B) \mid X_+ = [A \ B] [X_- \ U_-]\}.$$ 

Clearly, we have $(A_s, B_s) \in \Sigma_{(U_-, X)}$.

Suppose that we are interested in the system theoretic property $P$ of stabilizability. The corresponding set $\Sigma_P$ is then equal to $\Sigma_{\text{stab}} := \{(A, B) \mid (A, B) \text{ is stabilizable}\}$. Then, the data $(U_-, X)$ are informative for stabilizability if $\Sigma_{(U_-, X)} \subseteq \Sigma_{\text{stab}}$, that is, if all systems consistent with the input-state measurements are stabilizable.

Necessary and sufficient conditions for informativity for stabilizability were given in [3, Thm. 8] as follows.

**Theorem 1 (Data-driven Hautus test for stabilizability):**

The data $(U_-, X)$ are informative for stabilizability if and only if $\text{rank}(X_+ - \lambda X_-) = n$ for all $\lambda \in \mathbb{C}$ with $|\lambda| \geq 1$.

As mentioned before, there are situations in which we can conclude stabilizability from the data without being able to identify the true system uniquely. This is illustrated in the following example.

**Example 1:** Suppose that $n = 2$ and $m = 1$. Assume that we collect data on the time interval $[0, 2]$ and obtain

$$X = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad U_- = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$ 

Clearly, by Theorem 1 we see that these data are informative for stabilizability, as

$$\text{rank}(X_+ - \lambda X_-) = \text{rank} \begin{bmatrix} 1 & -\lambda \\ 0 & 1 \end{bmatrix} = 2 \quad \forall \lambda \in \mathbb{C}.$$ 

However, note that $\Sigma_{(U_-, X)}$ has infinitely many elements since $[X^T \ U_-^T]^T$ does not have full row rank.

#### B. Stabilizing feedback design from exact (noise free) data

Next, we will consider data driven stabilization by state feedback. In that context, for systems and data like in Section III-A, we take the control objective $O$: ‘interconnection with a state feedback controller yields a stable closed loop system’. The set of all systems that satisfy the control objective is then equal to $\Sigma_O = \{A \in \mathbb{R}^{n \times n} \mid A \text{ is stable}^1\}$.

For a given state feedback controller $K = K \in \mathbb{R}^{m \times n}$, the corresponding set of closed loop systems compatible with the data equals $\Sigma_{(U_-, X)}(K) = \{A + BK \mid (A, B) \in \Sigma_{(U_-, X)}\}$. The data $(U_-, X)$ are informative for stabilization by state feedback if $\Sigma_{(U_-, X)}(K) \subseteq \Sigma_O$, that is, if there exists a single controller $K$ such that $A + BK$ is stable for all $(A, B) \in \Sigma_{(U_-, X)}$.

Among several equivalent characterizations (see e.g. [3, Thm. 16 and Thm. 17] and [5, Thm. 2]), the following is the most appealing as it leads to a linear matrix inequality characterization where the size of unknowns does not depend on the size of the data.

**Theorem 2:** The data $(U_-, X)$ are informative for stabilizability by state feedback if and only if there exist $P = P^T > 0$, $L$, and $\beta > 0$ such that

$$\begin{bmatrix} P - \beta I & 0 & 0 \\ 0 & -P - L^T & 0 \\ 0 & -L & L \end{bmatrix} - \begin{bmatrix} X_+ & X_- \\ -X_+ & -X_- \end{bmatrix} \begin{bmatrix} X_+ \\ X_- \end{bmatrix}^T \geq 0.$$ 

$^1$We say that a matrix is stable if all its eigenvalues are contained in the open unit disk.
Moreover, if $P$ and $L$ satisfy the above linear matrix inequality for some $\beta$, then $K = LP^{-1}$ is a stabilizing feedback gain for all $(A, B) \in \Sigma_{(U_{-}, X)}$.

Again, there are situations in which we can design stabilizing feedback controllers from the data without being able to identify the true system uniquely as illustrated by the following example.

Example 2: Suppose the data given by

$$X = \begin{bmatrix} 1 & 0.5 & -0.25 \\ 0 & 1 & 1 \end{bmatrix}, \quad U_{-} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}. $$

One can conclude by Theorem 2 that the data $(U_{-}, X)$ are informative for stabilization by state feedback with $K = [-1 -0.5]$. It is worth noting that $\Sigma_{(U_{-}, X)}$ is not a singleton since $[X_{-}^\top \quad U_{-}^\top]^T$ does not have full row rank.

C. Stabilizing feedback design from noisy data

For given $n$ and $m$, consider the model class $\mathcal{M}$ consisting of all discrete-time linear input/state systems with unknown process noise of the form

$$x(t + 1) = Ax(t) + Bu(t) + w(t),$$

where $x \in \mathbb{R^n}$ is the state, $u \in \mathbb{R^m}$ is the control input and $w \in \mathbb{R^n}$ is an unknown noise term. Suppose we have data as in Section III-A given by $(U_{-}, X)$ that are collected from the the unknown system specified by the matrices $A_s$ and $B_s$. The noise $w$ is unknown, so $w(0), w(1), \ldots, w(T - 1)$ are not measured and hence are not part of the data. However, we do assume that we have the following information on the noise during the data sampling period: The matrix $W_{-} = [w(0) \quad w(1) \quad \cdots \quad w(T - 1)]$ satisfies the quadratic matrix inequality

$$\begin{bmatrix} I & W_{-}^\top \end{bmatrix}^T \Phi \begin{bmatrix} I \\ W_{-} \end{bmatrix} > 0$$

for all $(A, B) \in \Sigma_{(U_{-}, X)}$. We are interested in quadratic stabilization in the sense that we ask for a common Lyapunov matrix $P$ for all $(A, B) \in \Sigma_{(U_{-}, X)}$. Note that $P > 0$ satisfies (3) if and only if $Q := P^{-1}$ satisfies $Q - (A + BK)^T Q (A + BK) > 0$, which expresses that $V(x) = x^T Q x$ is a Lyapunov function for the system

$$x(t + 1) = (A + BK)x(t).$$

Observe that $(A, B) \in \Sigma_{(U_{-}, X)}$ if and only if $(A, B)$ satisfies the quadratic matrix inequality

$$\begin{bmatrix} I \\ A^T \end{bmatrix}^T \begin{bmatrix} I & X_+ \\ 0 & -X \end{bmatrix} \Phi_{11} \Phi_{12} \begin{bmatrix} I \\ 0 \end{bmatrix} \begin{bmatrix} I \\ W_{-} \end{bmatrix}^T \begin{bmatrix} I \\ A^T \end{bmatrix} > 0.$$

In addition, if we fix $K$ and $P$ then (3) is yet another quadratic matrix inequality in $(A, B)$:

$$\begin{bmatrix} I \\ A^T \end{bmatrix}^T \begin{bmatrix} P & 0 \\ 0 & -P \end{bmatrix} \Phi_{11} \Phi_{12} \begin{bmatrix} I \\ 0 \end{bmatrix} \begin{bmatrix} I \\ W_{-} \end{bmatrix}^T \begin{bmatrix} I \\ A^T \end{bmatrix} > 0.$$

Therefore, finding conditions for informativity for quadratic stabilization amounts to finding conditions under which the quadratic matrix inequality (5) holds for all $(A, B)$ satisfying the quadratic matrix inequality (4). In other words, checking informativity boils down to checking whether a given quadratic matrix inequality implies another. A generalization of the so-called $S$-lemma [7], [8] to matrix variables [6], [9] leads to necessary and sufficient conditions for this implication in terms of linear matrix inequalities.

Theorem 3: Assume that $\Phi_{22} < 0$, $\Phi_{11} - \Phi_{12} \Phi_{22}^{-1} \Phi_{21} \geq 0$ and $\text{rank} \begin{bmatrix} X_{-}^\top \\ U_{-}^\top \end{bmatrix}^T = n + m$. Define $\Theta := \Phi_{12} + X_{-} \Phi_{22}$. Then, the data $(U_{-}, X)$ are informative for quadratic stabilization if and only if there exists $P = P^T > 0$ satisfying

$$\begin{bmatrix} P & 0 \\ 0 & -P \end{bmatrix} \Phi_{11} \Phi_{12} \begin{bmatrix} I \\ 0 \end{bmatrix} \begin{bmatrix} I \\ W_{-} \end{bmatrix}^T \begin{bmatrix} P & 0 \\ 0 & -P \end{bmatrix} \Phi_{11} \Phi_{12} \begin{bmatrix} I \\ 0 \end{bmatrix} \begin{bmatrix} I \\ W_{-} \end{bmatrix}^T = \Theta > 0.$$

Moreover, if $P > 0$ satisfies (6) and (7) then

$$K = (U_{-}(\Phi_{22} + \Theta^T \Gamma^T \Theta)X_{-}^\top) \begin{bmatrix} X_{-} + \Phi_{22} + \Theta^T \Gamma^T X_{-}^\top \end{bmatrix}$$

is a stabilizing feedback gain for all $(A, B) \in \Sigma_{(U_{-}, X)}$, where $\Gamma = P^{-1} I X_{+} \Phi_{11} \Phi_{12} \begin{bmatrix} I \\ X_{+} \end{bmatrix}^T$ and $\bullet^\dagger$ denotes the Moore-Penrose pseudo-inverse.

D. Stabilization by using noisy input/output data

So far, we have illustrated the data informativity framework by using input/state data. Next, we will deal with the stabilization problem on the basis of input/output data.

As the model class $\mathcal{M}$, Consider input-output systems with noise represented by auto-regressive (AR) models of the form

$$y(t + L) + P_{L-1} y(t + L - 1) + \cdots + P_0 y(t) = Q_{L-1} u(t + L - 1) + \cdots + Q_0 u(t) + v(t).$$

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Here $L$ is a positive integer, called the order of the system. The input $u(t)$ and output $y(t)$ are assumed to take their values in $\mathbb{R}^p$ and $\mathbb{R}^q$, respectively. The term $v(t)$ represents unknown noise. The parameters of the model are real $p \times p$ matrices $P_0, P_1, \ldots, P_{L-1}$ and $p \times m$ matrices $Q_0, Q_1, \ldots, Q_L$.

We assume that we have noisy input-output data

$$u(0), u(1), \ldots, u(T), y(0), y(1), \ldots, y(T) \tag{9}$$

on a given time interval $[0, T]$ with $T \geq L$. These noisy data are obtained from the true system:

$$y(t + L) + P_{L-1}^t y(t + L - 1) + \cdots + P_0^t y(t) = Q_t^L u(t + L) + Q_{L-1}^t u(t + L - 1) + \cdots + Q_0^t u(t) + v(t). \tag{10}$$

As before, we assume that the unknown noise samples, captured in the matrix $V := [v(0) \; v(1) \; \cdots \; v(T - L)]$ satisfy the quadratic matrix inequality

$$\begin{bmatrix} I & V \end{bmatrix}^T \Pi \begin{bmatrix} I & V \end{bmatrix} \geq 0, \tag{10}$$

where $\Pi \in \mathbb{R}^{(p + T - L + 1) \times (p + T - L + 1)}$ is a known partitioned matrix

$$\Pi = \begin{bmatrix} \Pi_{11} & \Pi_{12} \\ \Pi_{21} & \Pi_{22} \end{bmatrix},$$

with $\Pi_{11} = \Pi_{11}^T \in \mathbb{R}^{p \times p}$ and $\Pi_{22} = \Pi_{22}^T \in \mathbb{R}^{(T - L + 1) \times (T - L + 1)}$. We assume that $\Pi_{22} < 0$ and $\Pi_{11} - \Pi_{12} \Pi_{22}^{-1} \Pi_{21} \geq 0$.

Denote $q := p + m$ and collect the (unknown) coefficient matrices $P_i$ and $Q_i$ in the $p \times (qL + m)$ matrix

$$R := [-Q_0 \; P_0 \; -Q_1 \; P_1 \; \cdots \; -Q_{L-1} \; P_{L-1}] \tag{11}$$

Also, arrange the data into the vectors $w(t) = [u(t)^T \; y(t)^T]^T$ for $t = 0, 1, \ldots, T$, and define the slightly adapted Hankel matrix $H(w)$ by

$$H(w) := \begin{bmatrix} w(0) & w(1) & \cdots & w(T - L) \\ \vdots & & & \vdots \\ w(L - 1) & w(L) & \cdots & w(T - 1) \\ y(L) & y(L + 1) & \cdots & y(T) \end{bmatrix}.$$ 

Partition

$$H(w) = \begin{bmatrix} H_1(w) \\ H_2(w) \end{bmatrix},$$

where $H_1'(w)$ contains the first $qL$ rows and $H_2'(w)$ the last $p$ rows. It is then easily verified that any input-output system in $\mathcal{M}$ for which

$$\begin{bmatrix} R & I \end{bmatrix} \begin{bmatrix} H_1(w) \\ H_2(w) \end{bmatrix} = V \tag{11}$$

for some $V$ satisfying (10) could have generated the noisy input-output data (9). Therefore, if $R$ satisfies (11) for some $V$ satisfying (10), we call the AR system corresponding to the matrix $R$ compatible with the data. Now define

$$N := \begin{bmatrix} I & H_2(w) \\ 0 & H_1(w) \end{bmatrix} \Pi \begin{bmatrix} I & H_2(w) \\ 0 & H_1(w) \end{bmatrix}^T. \tag{12}$$

Then by combining (10) and (11) we see that the system corresponding to the coefficient matrix $R$ is compatible with the data if and only if $R^T$ satisfies

$$\begin{bmatrix} I & R \end{bmatrix}^T N \begin{bmatrix} I & R \end{bmatrix} \geq 0. \tag{13}$$

To address the stabilization problem, we consider a feedback controller of the form

$$u(t + L) + G_L u(t + L - 1) + \cdots + G_0 u(t) = F_{L-1} y(t + L - 1) + \cdots + F_0 y(t). \tag{14}$$

Collect the coefficient matrices of $F_i$ and $G_i$ in the matrix

$$C := [G_0 \; G_1 \; \cdots \; G_{L-1} \; -F_{L-1}].$$

With the help of behavioral theory and in particular quadratic difference forms [10], one can state the following stability result for the closed loop system that is obtained by combining (8) and (14).

Lemma 1 ([11]): The closed loop system is stable if and only if there exists $\Psi = \Psi^T \in \mathbb{R}^{qL \times qL}$ such that $\Psi \succeq 0$ and

$$\begin{bmatrix} I_{qL} & -C \\ -R \end{bmatrix} \begin{bmatrix} 0_q & 0 \\ \Psi & 0 \end{bmatrix} \begin{bmatrix} I_{qL} \\ -C \\ -R \end{bmatrix} < 0. \tag{15}$$

Moreover, if $\Psi \succeq 0$ satisfies (15), then $\Psi > 0$. Motivated by this lemma, we now define the informativity for quadratic stabilization.

Definition 4: The noisy input-output data $u(0), u(1), \ldots, u(T), y(0), y(1), \ldots, y(T)$ are called informative for quadratic stabilization if there exist $C \in \mathbb{R}^{m \times qL}$ and $\Psi = \Psi^T \in \mathbb{R}^{qL \times qL}$ with $\Psi \succeq 0$ such that the QMI (15) holds for all $R$ that satisfy the QMI (13), with $N$ defined by (12).

As in Section III-C, finding conditions for the above informativity notion boils down to finding conditions under which the quadratic inequality (13) implies the one (15). Unlike the situation in Section III-C, however, (13) is in terms of the matrix $R^T$ whereas (15) is in terms of $R$. With the help of [6, Prop. 3.1], one can first reformulate the QMI (15) in terms of the variable $R^T$. Then, application of the matrix S-lemma [6, 9] leads to a linear matrix inequality characterization of data informativity for quadratic stabilization based on input-output data. To formulate this result, we first introduce the $2qL \times 2qL$ matrix

$$\tilde{N} = \begin{bmatrix} 0_{(qL-qL-p) \times (qL-p)} \\ 0 \\ N \end{bmatrix} = \begin{bmatrix} \tilde{N}_{11} & \tilde{N}_{12} \\ \tilde{N}_{21} & \tilde{N}_{22} \end{bmatrix}$$

where $\tilde{N}_{11}, \tilde{N}_{22} \in \mathbb{R}^{qL \times qL}$. Also, let

$$J = \begin{bmatrix} 0_{q \times (qL-1)} \\ I_{qL-1} \end{bmatrix}, \quad W = \begin{bmatrix} I_{qL-1} & 0 \\ 0 \\ 0_{m \times p} \end{bmatrix},$$

and $K = [J \; 0_{qL \times p} \; I_{qL}]$.

Theorem 4: Assume $H_1'(w)$ has full row rank. Then, the input-output data $u(0), u(1), \ldots, u(T), y(0), y(1), \ldots, y(T)$
are informative for quadratic stabilization if and only if there exists \( \Phi = \Phi^T \in \mathbb{R}^{qL \times qL} \) such that
\[
\Phi > \bar{N}_{11} - \bar{N}_{12}\bar{N}_{22}^{-1}\bar{N}_{21}
\]  
(16)
and
\[
\begin{bmatrix}
\bar{W}^T & 0 \\
0 & -\bar{N} & 0 \\
0 & 0 & -\Phi
\end{bmatrix}
> 0.
\]  
(17)

Moreover, if \( \Phi \) satisfies (16) and (17), then the controller with coefficient matrix \( C \) defined by (CONT) is a controller that stabilizes all systems compatible with the data.

IV. SUMMARY OF OTHER EXISTING RESULTS

Finally, we summarize further applications of the informativity framework. Various other problems have been studied, ranging from optimal control problems like linear quadratic regulation (LQR) and \( \mathcal{H}_\infty \) control, to the analysis of system properties like stability and dissipativity using noisy data. Table I provides an overview of the results. The second column of the table refers to the type of data that are used. Here, ‘E’ refers to exact data, and ‘N’ to noisy data. State, input-state, input-state-output and input-output are denoted by ‘S’, ‘IS’, ‘ISO’ and ‘IO’, respectively.

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TABLE I
SUMMARY OF RESULTS WITHIN THE INFORMATIVITY APPROACH.

REFERENCES


