Robust Stabilization in a Behavioral Framework

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Robust stabilization

Perturbed plant
(Nominal plant
+ Uncertainty)

w
Robust stabilization

Problem

Find a controller that stabilizes the nominal plant and all the plants in a specified neighborhood of the nominal plant.

Note: control by interconnection. The plant and controller share the variables on the terminals. No input output considerations, no signal flow, no feedback.
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Systems and behaviors

A dynamical system

\[ \Sigma = (T, W, \mathcal{B}) \]

- \( T \) is the set of independent variables: time, space, time and space
- \( W \) is the set of dependent variables, signal space
- \( \mathcal{B} \subseteq W^T := \{ w | w \text{ function } T \rightarrow W \} \): the behavior
  - the admissible trajectories

Linear differential systems

In this lecture, \( T = \mathbb{R} \) (‘1D systems’)
- \( W = \mathbb{R}^q \), \( w : \mathbb{R} \rightarrow \mathbb{R}^q \), \( (w_1(t), \ldots, w_q(t)) \)
- \( \mathcal{B} \) = solutions of a system of constant coefficient linear ODE’s.

For such \( \mathcal{B} \) there always existst \( R \in \mathbb{R}^{q \times q} \) such that

\[ \mathcal{B} := \{ w \in L^1_{loc}(\mathbb{R}, \mathbb{R}^q) | R \left( \frac{d}{dt} \right) w = 0 \} \]
Systems and behaviors

Notation

\[ \Sigma = (\mathbb{R}, \mathbb{R}^q, \mathcal{B}) \in \mathcal{L}^q, \text{ or: } \mathcal{B} \in \mathcal{L}^q \]
\[ \mathcal{B} = \text{ker}(R(\frac{d}{dt})). \] This is called a kernel representation of \( \mathcal{B} \).

Integer invariants associated with \( \mathcal{B} \):
output cardinality \( p(\mathcal{B}) := \) number of outputs of \( \mathcal{B} \),
input cardinality \( m(\mathcal{B}) := \) number of inputs of \( \mathcal{B} \).

\[ m(\mathcal{B}) + p(\mathcal{B}) = q. \]

Controllability

\( \mathcal{B} \in \mathcal{L}^q \) is called controllable if for all \( w_1, w_2 \in \mathcal{B} \) there exists \( T \geq 0 \) and \( w \in \mathcal{B} \) such that \( w(t) = w_1(t) \) for \( t < 0 \), and \( w(t) = w_2(t - T) \) for \( t \geq 0 \).

Notation: \( \mathcal{L}_{\text{cont}}^q \) subset of \( \mathcal{L}^q \) of controllable behaviors.
Rational representations

Neighborhood of a plant

How to formalize the concept of neighborhood of a given plant?
Rational representations

Neighborhood of a plant
How to formalize the concept of neighborhood of a given plant?

Rational kernel representations

Question
What do we mean by $R \left( \frac{d}{dt} \right) w = 0$ if $R(\xi)$ is rational?
Rational representations

Neighborhood of a plant

How to formalize the concept of neighborhood of a given plant?

Rational kernel representations

Question

What do we mean by \( R \left( \frac{d}{dt} \right) w = 0 \) if \( R(\xi) \) is rational?

J.C. Willems and Y. Yamamoto 2007:

Left coprime factorization

\( R = P^{-1}Q \) is called a left coprime factorization (LCPF) over \( \mathbb{R}[\xi] \) of \( R \) if

1. \( \det(P) \neq 0 \)
2. the matrix \( (P \ Q) \) is left prime over \( \mathbb{R}[\xi] \).
Rational representations

Meaning of $R\left(\frac{d}{dt}\right)w = 0$?

If $R = P^{-1}Q$ is LCPF, then $[R\left(\frac{d}{dt}\right)w = 0] \iff [Q\left(\frac{d}{dt}\right)w = 0]$.

Example

Let $\mathbb{B} := \{(w_1, w_2) \mid \frac{d}{dt}w_1 + \frac{1}{d}w_2 = 0\}$. Since

$R(\xi) = \begin{pmatrix} \xi & \frac{1}{\xi} \end{pmatrix} = \frac{1}{\xi}(\xi^2 \ 1) = P^{-1}Q$ is a LCPF

with $P = \xi$ and $Q = \begin{bmatrix} \xi^2 & 1 \end{bmatrix}$

by definition $\mathbb{B} = \{(w_1, w_2) \mid \frac{d^2}{dt^2}w_1 + w_2 = 0\}$.

Note

If $\mathbb{B}$ is controllable it admits a representation $R\left(\frac{d}{dt}\right)w = 0$, where $R$ is proper, stable (has all poles in $\mathbb{C}^-$), co-inner $(R(\xi)R^T(\xi) = I)$ and left prime (proper stable right-inverse).
Ball around the nominal plant

Let the nominal plant $\mathcal{P} \in \mathcal{L}_{\text{cont}}^q$ be represented by

$$\mathcal{P} = \{ w \mid R\left(\frac{d}{dt}\right)w = 0 \},$$

where $R$ is proper, stable, co-inner and left prime. For a given $\gamma > 0$ define the ball around $\mathcal{P}$ with radius $\gamma$:

$$B(\mathcal{P}, \gamma) := \{ \mathcal{P}_\Delta \in \mathcal{L}_{\text{cont}}^q \mid \exists \text{ a proper, stable, f.r.r. rational } R_\Delta \text{ such that } \mathcal{P}_\Delta = \ker(R_\Delta(\frac{d}{dt}))$$

and $\|R - R_\Delta\|_\infty \leq \gamma \}$. 

Lemma

All proper, stable, co-inner, left prime kernel representations $R\left(\frac{d}{dt}\right)w = 0$ of $\mathcal{P}$ yield the same ball $B(\mathcal{P}, \gamma)$. 
**Problem formulation**

A behavior $\mathcal{B} \in \mathcal{L}^q$ is called **stable** if for all $w \in \mathcal{B}$ we have $\lim_{t \to \infty} w(t) = 0$.

Recall: **output cardinality** of $\mathcal{B} \in \mathcal{L}^q$: $p(\mathcal{B}) :=$ the number of outputs of $\mathcal{B}$.

**Full interconnection** of $\mathcal{P} \in \mathcal{L}^q$ and $\mathcal{C} \in \mathcal{L}^q$: the intersection $\mathcal{P} \cap \mathcal{C}$.

The interconnection is called **regular** if $p(\mathcal{P} \cap \mathcal{C}) = p(\mathcal{P}) + p(\mathcal{C})$.

**Robust stabilization problem**

Given a radius $\gamma > 0$, find a controller $\mathcal{C} \in \mathcal{L}^q$ such that $\mathcal{P}_\Delta \cap \mathcal{C}$ is stable and a regular interconnection for all $\mathcal{P}_\Delta \in B(\mathcal{P}, \gamma)$.

**Optimal robust stabilization**

Find

$$\gamma^* := \sup \{ \gamma > 0 \mid \exists \mathcal{C} \in \mathcal{L}^q : \mathcal{P}_\Delta \cap \mathcal{C} \text{ is stable and a regular interconnection for all } \mathcal{P}_\Delta \in B(\mathcal{P}, \gamma) \}.$$
Dissipative linear differential systems

Dissipativity

Let $\mathcal{B} \in \mathcal{L}^q_{\text{cont}}, \Sigma = \Sigma^T \in \mathbb{R}^{q \times q}$. The quadratic form $\mathbf{w}^T \Sigma \mathbf{w}$ is called a supply rate.

$\mathcal{B}$ is called $\Sigma$-dissipative if
\[ \int_{-\infty}^{+\infty} \mathbf{w}^T \Sigma \mathbf{w} \, dt \geq 0 \]
for all $\mathbf{w} \in \mathcal{B} \cap \mathcal{D}$.

$\mathcal{B}$ is said to be strictly $\Sigma$-dissipative if there exists an $\epsilon > 0$ such that $\mathcal{B}$ is $(\Sigma - \epsilon I)$-dissipative.

$\mathcal{B}$ is $\Sigma$-dissipative if and only if there exists a QDF $Q_{\psi}$ (a storage function) such that
\[ \frac{d}{dt} Q_{\psi}(\mathbf{w}) \leq \mathbf{w}^T \Sigma \mathbf{w} \quad \forall \mathbf{w} \in \mathcal{B}. \]

Storage functions are quadratic functions of the state of $\mathcal{B}$: given a minimal state $x$ for $\mathcal{B}$ there exists $K = K^T \in \mathbb{R}^{n \times n}$ such that $Q_{\psi}(\mathbf{w}) = x^T K x$.

$Q_{\psi}$ is called positive definite (negative definite) if $K > 0$ ($K < 0$).
Dissipativity synthesis problem

Let $\mathcal{P}_{\text{full}} \in \mathcal{L}^{w+v+c}$, $\mathcal{C} \in \mathcal{L}^{c}$. Then we define the interconnection through $c$ as

$$\mathcal{P}_{\text{full}} \wedge_c \mathcal{C} := \{(w, v, c) \mid (w, v, c) \in \mathcal{P}_{\text{full}} \text{ and } c \in \mathcal{C}\}.$$  

Definitions

- A controller $\mathcal{C}$ is called regular if the interconnection of $\mathcal{P}_{\text{full}}$ and $\mathcal{C}$ is regular, i.e. $p(\mathcal{P}_{\text{full}} \wedge_c \mathcal{C}) = p(\mathcal{P}_{\text{full}}) + p(\mathcal{C})$
- A controller $\mathcal{C}$ is called disturbance free if $v$ is free in $\mathcal{P}_{\text{full}} \wedge_c \mathcal{C}$
- A controller $\mathcal{C}$ is called stabilizing if it is disturbance free and
  $$[(w, 0, c) \in \mathcal{P}_{\text{full}} \wedge_c \mathcal{C}] \Rightarrow \lim_{t \to \infty} w(t) = 0.$$  
- For $\gamma > 0$, a controller $\mathcal{C} \in \mathcal{L}^{c}$ is called strictly $\gamma$-contracting if $\exists \epsilon > 0$ such that $\forall (w, v) \in (\mathcal{P}_{\text{full}} \wedge_c \mathcal{C})_{(w,v)} \cap \mathcal{L}^{2}$ we have
  $$\|w\|_2 \leq \left(\frac{1}{\gamma} - \epsilon\right)\|v\|_2.$$  

Such controller $\mathcal{C}$ is said to be a solution to the dissipativity synthesis problem.
Solution to the dissipativity synthesis problem

For a given behavior $\mathcal{B} \in \mathcal{L}_\text{cont}^q$ define

$$\mathcal{B}^\perp := \{ w \mid \int_{\mathbb{R}} w^\top w' = 0 \ \forall w' \in \mathcal{B} \cap \mathcal{D} \}$$

Notation: $(\mathcal{P}_\text{full})_{(w,v)} := \{ (w,v) \mid \exists c : (w,v,c) \in \mathcal{P}_\text{full} \}$.

Let $\Sigma_\gamma := \begin{pmatrix} -I & 0 \\ 0 & \frac{1}{\gamma^2}I \end{pmatrix}$.

Theorem (Trentelman and Willems 1999):

Let $\mathcal{P}_\text{full} \in \mathcal{L}_u^{q+v+c}$ be controllable. Assume $v$ is free, $(w,v)$ is detectable from $c$, and $c$ is observable from $(w,v)$. Let $\gamma > 0$. There exists a solution $\mathcal{C}$ to the dissipativity synthesis problem if and only if $(\mathcal{P}_\text{full})_{(w,v)}^\perp$ is $-\Sigma_\gamma^{-1}$-dissipative and has a negative definite storage function.
Small gain argument

Back to the robust stabilization problem:
Again, let $\mathcal{P} = \{ w \mid R(\frac{d}{dt})w = 0 \}$ be the nominal plant, with $R$ proper, stable and co-inner.

$$\mathcal{P}_{aux} := \{ (w, v, c) \mid R(\frac{d}{dt})w + v = 0, \ c = w \}.$$ Let $R(\xi) = P^{-1}(\xi)Q(\xi)$ be a LCPF with $P$ Hurwitz. Then by definition

$$\mathcal{P}_{aux} = \{ (w, v, c) \mid Q(\frac{d}{dt})w + P(\frac{d}{dt})v = 0, \ c = w \}.$$ Small gain theorem

Let $\gamma > 0$. Let $C \in \mathcal{L}^q$. Then $\mathcal{P}_\Delta \cap C$ is stable and a regular interconnection for all $\mathcal{P}_\Delta \in B(\mathcal{P}, \gamma)$ if and only if $C$ is a solution to the dissipativity synthesis problem for $\mathcal{P}_{aux}$. 
Solution to the robust stabilization problem

As

- $v$ is free in $\mathcal{P}_{\text{aux}}$ (since $Q$ has a full row rank)
- $c$ is observable from $(w, v)$ in $\mathcal{P}_{\text{aux}}$
- $(w, v)$ is detectable from $c$

Theorem

Let $\gamma > 0$. There exists a controller $C \in \mathcal{L}^q$ such that $\mathcal{P}_\Delta \cap C$ is stable and a regular interconnection for all $\mathcal{P}_\Delta \in B(\mathcal{P}, \gamma)$ if and only if $(\mathcal{P}_{\text{aux}})_{(w,v)}^\perp$ is strictly $-\Sigma_{\gamma}^{-1}$ dissipative and has a negative definite storage function.

Computation

Condition for existence is representation free. Computation of required $C$ from a kernel or image representation of $\mathcal{P}_{\text{full}}$ involves polynomial spectral factorization.
Optimal robust stabilization

Problem

Find

\[ \gamma^* := \sup \{ \gamma > 0 \mid \exists C \in \mathcal{L}^q : \mathcal{P}_\Delta \cap C \text{ is stable and a regular interconnection for all } \mathcal{P}_\Delta \in B(\mathcal{P}, \gamma) \}. \]

By the previous theorem: \( \gamma^* \) is the supremum over all \( \gamma > 0 \) such that \((\mathcal{P}_{\text{aux}})_{(w,v)}^\perp\) is strictly \(-\Sigma_{\gamma}^{-1}\) dissipative and has a negative definite storage function.

It can be shown: \((\mathcal{P}_{\text{aux}})_{(w,v)}^\perp\) is strictly \(-\Sigma_{\gamma}^{-1}\) dissipative if and only if \(0 < \gamma < 1\).

So: the real issue to compute the supremum over all \(0 < \gamma < 1\) for which the smallest storage function is negative definite.
Solution to the optimal robust stabilization problem

Recall: \( P \) is the nominal plant. Consider the system \( P^\perp \) with manifest variable \( \tilde{w} \). \( P^\perp \) is (trivially) strictly dissipative w.r.t \( \| \tilde{w} \|^2 \).

Let the maximal and minimal storage functions be given by \( \Psi^+(\zeta, \eta) \) and \( \Psi^-(\zeta, \eta) \) respectively. These can be computed by means of polynomial spectral factorization.

The smallest storage function of \( (P_{\text{aux}})^\perp \) as a \( -\Sigma^{-1} \)-dissipative system is given by the two-variable polynomial matrix \( \Psi_{\gamma} = (1 - \gamma^2)\Psi_- + \gamma^2\Psi_+ \).

Coefficient matrices of \( \Psi_+(\zeta, \eta) \) and \( \Psi_-(\zeta, \eta) \): \( \tilde{\Psi}_- \) and \( \tilde{\Psi}_+ \).

Theorem

\[
\gamma^* = \sqrt{\frac{\lambda_{\text{max}}(\tilde{\Psi}_- \tilde{\Psi}_+^\perp)}{\lambda_{\text{max}}(\tilde{\Psi}_- \tilde{\Psi}_+^\perp) - 1}}.
\]

In particular, for \( \gamma > 0 \) the following holds: there exists \( C \in \mathcal{L}^q \) such that \( P_\Delta \cap C \) is stable for all \( P_\Delta \in B(P, \gamma) \) if and only if \( \gamma < \gamma^* \). \( \tilde{\Psi}_+^\perp \) is the Moore-Penrose matrix inverse of \( \tilde{\Psi}_+ \).
Concluding remarks

1. Representation free conditions for the existence of a robustly stabilizing controller were obtained.

2. Optimal $\gamma^*$ was obtained using storage functions of $\mathcal{P}^\perp$.

3. Extending the results to:
   - different perturbations, for example nominal plant is given by $w = M(\frac{d}{dt})\ell$ and perturbed plant is given by $w = (M + \Delta)\ell$,
   - connection with gap metric between behaviors,
   - the partial interconnection case.

THANKS FOR YOUR ATTENTION