SENSE AND SIMPLICITY:

a talk on two matrices

on the occasion of Malo Hautus’ afscheidsrede

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University of Groningen, The Netherlands
Controllability and the Hautus test
Controllability

Linear dynamical system with inputs:

\[ \dot{x}(t) = Ax(t) + Bu(t) \]

\( x(t) \in \mathbb{R}^n \), the state, \( u(t) \in \mathbb{R}^m \), control input, \( A \in M_{n \times n} (\mathbb{R}) \), \( B \in M_{n \times m}(\mathbb{R}) \).
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The system is called **controllable** if for every pair of states \( x_0, x_1 \) there exists \( T > 0 \) and an input function \( u \) on \([0, T]\) such that \( x(0) = x_0 \) and \( x(T) = x_1 \).
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We say: the pair \((A, B)\) is controllable. *Kalman, 1963*
Test for controllability

\[ n \times nm \text{ matrix } (B, AB, A^2B, \ldots, A^{n-1}B) \]
Test for controllability

\( n \times nm \) matrix \( (B, AB, A^2B, \ldots, A^{n-1}B) \)

**Theorem (Kalman 1963):** \((A, B)\) is controllable if and only if \((B, AB, A^2B, \ldots, A^{n-1}B)\) has rank \(n\).
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**Theorem (Kalman 1963):** \((A, B)\) is controllable if and only if \((B, AB, A^2B, \ldots, A^{n-1}B)\) has rank \(n\).

\[ \text{im}(B, AB, A^2B, \ldots, A^{n-1}B) \subseteq \mathbb{R}^n \] is equal to the the reachable subspace

\[ \{ x_1 \in \mathbb{R}^n \mid \exists u \text{ such that } x(0) = 0 \text{ and } x_1 = x(T) \} \]
The Hautus test


Main result:
The Hautus test


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Theorem (Hautus 1969): $(A, B)$ is controllable if and only if the $\begin{bmatrix} n 	imes (n + m) \end{bmatrix}$ matrix $(A - \lambda I, B)$ has rank $n$ for every eigenvalue $\lambda$ of $A$. 
The Hautus test


Main result:

**Theorem (Hautus 1969):** $(A, B)$ is controllable if and only if the $n \times (n + m)$ matrix $(A - \lambda I, B)$ has rank $n$ for every eigenvalue $\lambda$ of $A$.

Equivalently: there is no left eigenvector of $A$ that is a left zero vector of $B$. 
Proving the Hautus test

Standard result in every introductory course to linear systems.

(⇒)
Proving the Hautus test

Standard result in every introductory course to linear systems.

\[(\Leftrightarrow)\]

\[
\begin{align*}
\text{rank}(A - \lambda I, B) < n & \implies \\
\exists \eta \neq 0 \text{ such that } \eta A = \lambda \eta, \eta B = 0 & \implies \\
\exists \eta \neq 0 \text{ such that } \eta(B, AB, A^2B, \ldots, A^{n-1}B) = 0 & \implies \\
\text{rank}(B, AB, A^2B, \ldots, A^{n-1}B) < n & \implies \\
(A, B) \text{ not controllable}
\end{align*}
\]
Proving the Hautus test

(⇔)
(⇐)

Up to now (for students in Groningen):
assume \((A, B)\) not controllable. Then the reachable subspace is a proper \(A\)-invariant subspace of \(\mathbb{R}^n\) containing \(\text{im}(B)\)
Proving the Hautus test

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Write matrices for \(A\) and \(B\) with respect to a basis adapted to this subspace.
This yields a contradiction with \((A - \lambda I, B)\) full row rank for all \(\lambda\).
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In the original paper from 1969 one can find Malo’s original proof:
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Sense and Simplicity!
Hautus’ proof of the Hautus test

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Then \(\deg(\psi) \geq 1\)
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Define \(\zeta := \eta\phi(A)\). Note: \(\zeta \neq 0!\)
Then \(\zeta A = \lambda \zeta\). Also \(\zeta B = \eta\phi(A)B = 0\). Contradiction!
Hautus’ proof of the Hautus test

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I found several other interesting results with equally nice proofs!

I will discuss one of these because it will be useful in the pole placement problem later on.
Reduction of the number of inputs

Again we look at the system

\[ \dot{x}(t) = Ax(t) + Bu(t) \]

\( x(t) \in \mathbb{R}^n \), the state space, \( u(t) \in \mathbb{R}^m \), control input, \( A \in M_{n \times n}(\mathbb{R}), B \in M_{n \times m}(\mathbb{R}) \). Note: \( m \) is the number of inputs.
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**Question:** can we reduce the number of inputs so that the system remains controllable?
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**Question:** can we reduce the number of inputs so that the system remains controllable?

Does there exists \( C \in M_{m \times m'}(\mathbb{R}) \) with \( m' < m \) such that the system

\[ \dot{x}(t) = Ax(t) + BCv(t) \]

with \( v(t) \in \mathbb{R}^{m'} \) is still controllable?
Reduction of the number of inputs

Let $\lambda$ be an eigenvalue of $A$

$\omega(\lambda) := \text{the maximal number of eigenvectors with eigenvalue } \lambda$

$\omega(\lambda) = n - \text{rank}(A - \lambda I)$

$\omega(A) := \max_{\lambda} \omega(\lambda)$
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**Theorem:** If $(A, B)$ is controllable then there exists a $m \times \omega(A)$ matrix $C$ such that $(A, BC)$ is controllable.
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The number of inputs can be reduced to $\omega(A)$

**Example:** if $A$ has $n$ distinct eigenvalues then $\omega(A) = 1$. 
Reduction of the number of inputs

How to find such $m \times \omega(A)$ matrix $C'$? Use the Hautus test!
Reduction of the number of inputs

How to find such $m \times \omega(A)$ matrix $C$? Use the Hautus test!
Let $\lambda_1, \lambda_2, \ldots, \lambda_\nu$ be the distinct eigenvalues of $A$.
rank$(A - \lambda_k I, B) = n$
rank$(A - \lambda_k I) = n - \omega(\lambda_k)$
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\[
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\]
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\text{rank}(A - \lambda_k I) = n - \omega(\lambda_k)
\]

There must be $\omega(\lambda_k)$ columns of $B$ that together with $n - \omega(\lambda_k)$ columns of $A - \lambda_k I$ form a basis of $\mathbb{R}^n$.

Hence: \exists a $m \times \omega(A)$ matrix $C_k$ (only 1’s and 0’s) such that

\[
\text{rank}(A - \lambda_k I, BC_k) = n \quad (k = 1, 2 \ldots \nu)
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Reduction of the number of inputs

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$$\text{rank}(A - \lambda_k I, BC_k) = n \quad (k = 1, 2 \ldots \nu)$$

Parameter vector $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\nu)$
Define $C(\alpha) = \alpha_1 C_1 + \alpha_2 C_2 + \ldots + \alpha_\nu C_\nu$
Reduction of the number of inputs

How to find such \( m \times \omega(A) \) matrix \( C \)? Use the \textbf{Hautus test}!

Let \( \lambda_1, \lambda_2, \ldots, \lambda_\nu \) be the distinct eigenvalues of \( A \).

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\text{rank}(A - \lambda_k I, B) = n \\
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Parameter vector \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\nu) \)

Define \( C(\alpha) = \alpha_1 C_1 + \alpha_2 C_2 + \ldots + \alpha_\nu C_\nu \)

The set of \( \alpha \)'s for which \( (A, BC(\alpha)) \) is \textbf{not} controllable is the union of \( n \) proper algebraic varieties.
Hautus and the pole placement problem
The stabilization problem

Linear system: \[ \dot{x}(t) = Ax(t) + Bu(t) \]
\[ x(t) \in \mathbb{R}^n, u(t) \in \mathbb{R}^m, A \in M_{n \times n}(\mathbb{R}), B \in M_{n \times m}(\mathbb{R}) \]
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State feedback \[ u(t) = Fx(t), F \in M_{m\times n}(\mathbb{R}) \]
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If such \( F \) exists then \( (A, B) \) is called stabilizable
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If such $F$ exists then $(A, B)$ is called stabilizable

Theorem (Hautus test for stabilizability): $(A, B)$ is stabilizable if and only if $(A - \lambda I, B)$ has rank $n$ for every eigenvalue $\lambda$ of $A$ with $\Re(\lambda) \geq 0$
The pole placement problem

System \( \dot{x}(t) = Ax(t) + Bu(t) \), state feedback \( u(t) = Fx(t) \)
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System $\dot{x}(t) = Ax(t) + Bu(t)$, state feedback $u(t) = Fx(t)$

Eigenvalues of $A + BF$ are the roots of the characteristic polynomial $\chi_{A+BF}$ of $A + BF$.

**Question:** what are conditions on $A$ and $B$ such that for every monic, real polynomial $p$ of degree $n$ there exists $F \in M_{m \times n}(\mathbb{R})$ such that $\chi_{A+BF} = p$?
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**Theorem:** For every polynomial
\[
p(z) = z^n + p_{n-1}z^{n-1} + \ldots + p_1z + p_0 \text{ with real coefficients there exists } F \in M_{m \times n}(\mathbb{R}) \text{ such that } \chi_{A+BF} = p
\]
if and only if \( (A, B) \) is controllable.
Proving the pole placement theorem

In this audience, many people teach or have taught this to their students. The most common proof is as follows:
Proving the pole placement theorem

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\[(\Rightarrow)\]

\((A, B)\) not controllable \(\Rightarrow\)
\[\exists \lambda\text{ and } \eta \neq 0\text{ such that } \eta A = \lambda \eta \text{ and } \eta B = 0 \Rightarrow\]
For every \(F\) we have \(\eta(A + BF) = \lambda \eta \Rightarrow\)
For every \(F\), \(\lambda\) is an eigenvalue of \(A + BF\) \(\Rightarrow\)
there exists a polynomial \(p\) with \(\chi_{A+BF} \neq p\) for all \(F\) (take any polynomial such that \(p(\lambda) \neq 0\))
Proving the pole placement theorem

(⇐)
Proving the pole placement theorem

\[(\Longleftrightarrow)\]

For the converse implication we need to construct for every monic, real polynomial \(p\) of degree \(n\) a \(F \in M_{m \times n}(\mathbb{R})\) such that
\[\chi A + BF = p\]
Proving the pole placement theorem

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\[\chi A + BF = p\]


**Heymann’s lemma:** Let \((A, B)\) be controllable and let \(b_1, b_2, \ldots, b_m\) be the column vectors of \(B\). Then for any \(i = 1, 2, \ldots, m\) there exists \(F_i \in M_{m \times n}(\mathbb{R})\) such that \((A + BF_i, b_i)\) is controllable
Proving the pole placement theorem

Heymann’s lemma reduces the problem to the controllable case with one input. This is an easy problem:
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With respect to a suitable basis, $A + BF_i$ is in ‘companion form’ and $b_i = (0, 0, \ldots, 1)^T$. 
Proving the pole placement theorem

Heymann’s lemma reduces the problem to the controllable case with one input. This is an easy problem:

With respect to a suitable basis, \( A + BF_i \) is in ‘companion form’ and \( b_i = (0, 0, \ldots, 1)^T \).

Less known: in the paper by Malo Hautus, ‘Stabilization, controllability and observability of linear autonomous systems’, Proceedings Koninklijke Nederlandse Academie voor Wetenschappen, series A, 1970, there is a completely different proof, that does not use Heymann’s lemma!
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I would like to take a look at that proof here: history of Systems Theory!
A theorem of R. Rado


**Theorem:** If $\Sigma = \{S_1, S_2, \ldots, S_n\}$ is a collection of subsets of a vector space $V$ such that for $k = 1, 2, \ldots, n$ the union of each $k$-tuple of sets in $\Sigma$ contains at least $k$ independent vectors, then there exists a set of independent vectors $\{x_1, x_2, \ldots, x_n\}$ in $V$ such that $x_k \in S_k$ $(k = 1, 2 \ldots, n)$. 
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**Corollary (finite dimensional case):** If \( \{ M_1, M_2, \ldots, M_n \} \) is a collection of matrices with \( n \) rows such that 
\[
\text{rank}(M_{i_1}, M_{i_2}, \ldots, M_{i_k}) \geq k
\]
for each choice of mutually distinct \( i_1, i_2, \ldots, i_k \), then there exists a basis \( \{ x_1, x_2, \ldots, x_n \} \) of \( \mathbb{R}^n \) with \( x_i \in \text{im}(M_i) \) \((i = 1, 2, \ldots, n)\).
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**Corollary (finite dimensional case):** If $\{M_1, M_2, \ldots, M_n\}$ is a collection of matrices with $n$ rows such that $\text{rank}(M_{i_1}, M_{i_2}, \ldots, M_{i_k}) \geq k$ for each choice of mutually distinct $i_1, i_2, \ldots, i_k$, then there exists a basis $\{x_1, x_2, \ldots, x_n\}$ of $\mathbb{R}^n$ with $x_i \in \text{im}(M_i)$ ($i = 1, 2, \ldots, n$).
Hautus’ proof of the pole placement theorem

**Lemma (Hautus):** Let \((A, B)\) be controllable, and let \(\mu_1, \mu_2, \ldots, \mu_k\) be \(k\) distinct numbers \((1 \leq k \leq n)\) such that none of the \(\mu_i\) is an eigenvalue of \(A\). Define \(M_i \in M_{n \times m}(\mathbb{R})\) by

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M_i = (A - \mu_i I)^{-1} B \quad (i = 1, 2 \ldots k)
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Then \(\text{rank}(M_1, M_2, \ldots, M_k) \geq k\).
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Then $(A + BF_0)x_i = \lambda_i x_i$ ($i = 1, 2, \ldots, n$) so $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of $A + BF_0$ (distinct eigenvalues!)
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**Note:** already very close to pole placement: every polynomial \(p\) of the form \(p(z) = (z - \lambda_1)(z - \lambda_2) \cdots (z - \lambda_n)\) with \(\lambda_i \neq \lambda_j\) real and \(\lambda_i\) not an eigenvalue of \(A\) can be assigned.
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Note that therefore \(\omega(A + BF_0) = 1\).

Also \((A + BF_0, B)\) is still controllable.

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More evidence for this is the book ‘Control Theory for Linear Systems’ by H.L. Trentelman, A.A. Stoorvogel and M.L.J. Hautus unfortunately sold out....
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