Computing potential flows around Joukowski airfoils using FFTs

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Bachelor thesis

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Chapter 1

Introduction

In this bachelor thesis we are going to discuss 2-dimentional potential airflows around so called Joukowski airfoils. The reason for doing this is an assignment for the course Computational methods of science, also see [1]. In this assignment (discussed in section 3.1.1) we determine the speed and pressure around a cylinder using FFT's. When we know the pressure we can say something about the net (lift) force acting on the body.

As you can image, a cylinder is not a very interesting model. So the goal of this thesis is to find a way to transform this flow around the cylinder onto a profile that is more interesting.

We do this by using the Joukowski transformation which maps a cylinder on an airfoil shaped body, the so called Joukowski airfoil. Before we can transform the speed around the cylinder we must first determine the speed around a cylinder with circulation. We have to do this in order to satisfy the so called Kutta-Joukowski condition. This condition states that the flow should leave the body smoothly. We determine the speed around a cylinder with circulation with the help of the Hess and Smith method combined with FFT's. The Hess and Smith method is a boundary integral method for flows around arbitrary shapes, that uses sources and circulations to create a flow. As a check we also solve the problem using super positioning of singularities.

After this we transform the flow to a flow around the Joukowski airfoil in such a way that it is physically realistic. In the last section we will discuss the most interesting results obtained.

Chapter 2

Potential and Bernoulli

The fact that we are using potential flows means that we're only going to consider flows in perfect fluids without rotation. This fact means that there is a speed potential Φ such that the speed V of the flow is given by $V = \nabla\Phi$. So $V = (u, v) = (\frac{d\Phi}{dx}, \frac{d\Phi}{dy})$. Together with the fact that we have mass conservation, $\nabla \cdot V = 0$, this leads to to the 2-dimensional potential equation:

$$\Delta\Phi = \Phi_{xx} + \Phi_{yy} = 0$$

The great advantage of potential flows is that we can calculate the velocity field of the flow without knowing the pressure. So we can first calculate the velocity at a certain point by solving the potential equation and then calculate the pressure at these points by using the law of Bernoulli, which gives a relation between the speed and the pressure:

$$\frac{1}{2}\rho|v|^2 + p = c$$

In this equation ρ is the density of the air, v is the speed, and p is the pressure.

Since we have no rotations we have $\nabla \times V = 0$, or $u_y - v_x = 0$. We now introduce Ψ which is known as the stream function of Stokes. We define Ψ by $u = \Psi_y$ en $v = -\Psi_x$. From the lack of rotation we have that $\Psi_{xx} + \Psi_{yy} = 0$, so we see that Ψ is also a potential function. If we take a closer look we see that when Ψ is constant it defines the streamlines of the velocity field:

$$\Psi_x + \Psi_y \frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{\Psi_x}{\Psi_y} = \frac{v}{u}$$

We also see that Φ and Ψ follow the Cauchy-Riemann equations:

$$\Phi_x = \Psi_y$$

$$\Phi_y = -\Psi_x$$

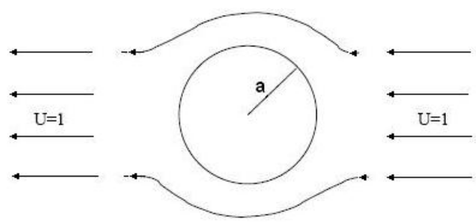
This is an essential condition for analyticity. This means that if the complex speed potential $\chi = \Phi + i\Psi$ is an analytic function of $z = x + iy$, then Φ and Ψ follow the above Cauchy-Riemann equations. If we call $\chi'(z) = \omega(z)$ the complex speed then $\chi(z)$ is analitic if $\omega(z)$ is finite, and thus the potential equations are solved.

Chapter 3

Flow around a cylinder

3.1 Flow around a cylinder without circulation

3.1.1 Boundary integral method



To understand the air flow around airfoils we will start to examine a simpler flow, namely the flow around a 2-dimentional cylinder with radius a. In order to solve this problem we will use a boundary integral method (BIM) or panel method as seen in [1]. This is the original assignment of the course Computational methods of science on which this thesis is based.

Consider a flow around an infinite long cylinder in a perfect fluid without rotation. This means that there exists a potential $V = \nabla\Phi$ which we will use to calculate the velocity at the profile and thus, using Bernoulli's law $\frac{1}{2}\rho|v|^2 + p = c$, we can determine the pressure on the profile. Assume that the cylinder is placed in an uniform flow $V = (-1, 0)$. Since $\rho = 1$ we can say that we can determine c for the initial conditions $|v| = 1$ and $p = 0$. So $c = \frac{1}{2}$.

We now need to give some boundary condition, so we can solve the potential equation $\Delta\Phi = 0$. Physics demand from us that no mass goes trough the profile. In other words the speed normal to the profile should be zero:

$$\frac{\partial\Phi}{\partial n} = 0$$

At infinity the velocity goes to a horizontal flow given by $V = (-1, 0)$ and thus the potential goes to x. In order to avoid problems with this condition we introduce a disturbance potential $\zeta = \Phi + x$. If we transform the boundary conditions to this potential we get:

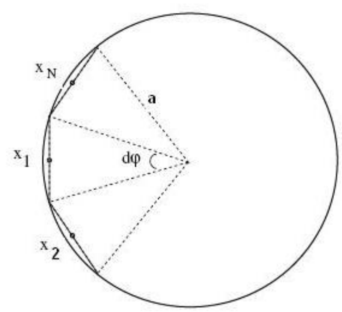
In order to solve this first we are going to change to polar coordinates, $(x, y) = (r\cos(\theta), r\sin(\theta))$.

For the BIM we need to divide our boundary into N straight line segments Γ_j, $j = 1, ..., N$. On the centre of these segments we have the points, called nodes, at which we are going to

$$\begin{array}{cc} \Delta\zeta = 0 & x^2+y^2>a^2 \\ \frac{\partial\Phi}{\partial n} = \frac{\partial\zeta}{\partial n} + \frac{\partial x}{\partial n} = \frac{\partial x}{\partial n} = n_1 & x^2+y^2=a^2 \\ \zeta, \frac{\partial\zeta}{\partial n} \to 0 & x^2+y^2 \to \infty \end{array}$$

$$\begin{array}{cc} \Delta\zeta = 0 & r>a \\ \frac{\delta\zeta}{\delta n} = \cos(\theta) & r=a \\ \zeta, \frac{\delta\zeta}{\delta n} \to 0 & r \to \infty \end{array}$$

evaluate the potential.



We will need a suitable form of the Laplace equation in order to use the BIM. From [1] we know that we can write the Laplace equation as follow

$$\frac{1}{2}\Phi_i + \sum_{j=1}^{n} \tilde{H}_{i,j}\Phi_j = \sum_{j=1}^{N} G_{i,j}q_j \qquad i = 1,..,N$$

in which u_i is the velocity in node i , $q_i = \frac{\partial\Phi}{\partial n}\mid_i$ is the velocity in the normal direction in the point i, and H and G are given by

$$\tilde{H}_{ij} = \int_{\Gamma_j} q^* d\Gamma \qquad G_{ij} = \int_{\Gamma_j} \Phi^* d\Gamma$$

Here $\Phi^*$and q^* are the given boundary conditions. We can now calculate $H_{i,j}$and $G_{i,j}$ using the midpoint rule:

$$\tilde{H}_{i,j} = \int_{\Gamma_j} q^* d\Gamma = q_j^* \Delta\Gamma_j$$

$$G_{i,j} = \int_{\Gamma_j} \Phi^* d\Gamma = \Phi_j^* \Delta\Gamma_j$$

Here $\Delta\Gamma_j$ is the length of the j line segment. Since we are looking at a 2-dimensional problem $\Phi_j^* = \frac{1}{2\pi}\ln(\frac{1}{r_{i,j}})$ in which $r_{i,j)}$ is the distance between the nodes i and j. If we write

$$H_{ij} = \begin{cases} \tilde{H}_{ij} \ i \neq j \\ \frac{1}{2} \ i = j \end{cases}$$

We can say that $H\Phi = Gq$. Because of the symmetry of the problem H and G are circulant matrices. We can now solve the system $H\Phi = GQ$, but because of the symmetry we can write this as a periodic convolution $(h * \Phi) = (H\Phi)_i$ and $(g * Q)_i = (GQ)_i$, in which g and h are the first rows of respectively G and H. Now we use Fourier transforms to solve these equations. The results are displayed in 3.1.

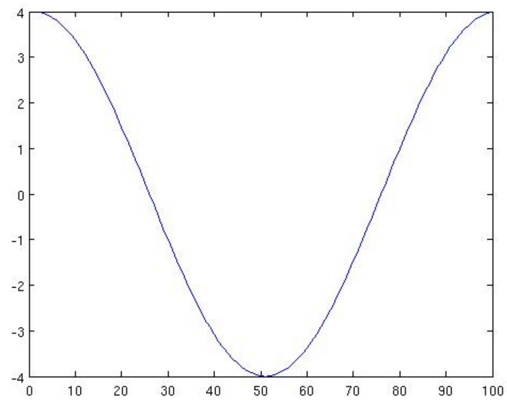


Figure 3.1: Potential on the circle

We have found the potential on the boundary, so we can calculate the air velocity at the surface. We have that $V = \Delta\Phi$,and since the velocity normal to the surface is zero, V is equal to the tangential velocity. So we can estimate the velocity by saying:

$$V_j \approx \frac{\Phi_{j+1} - \Phi_j}{\Delta\Gamma_j}$$

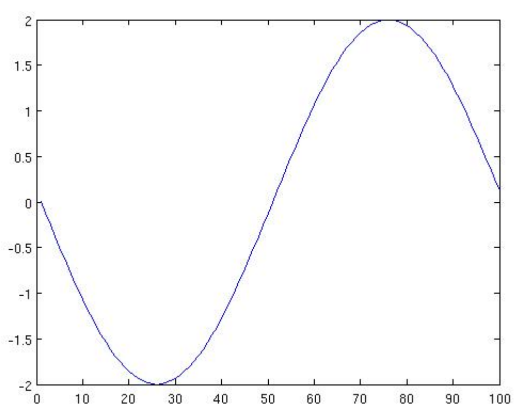


Figure 3.2: Tangential velocity on the circle (no circulation)

Which is depicted in 3.2.With this we can also calculate the pressure at the surface, as seen in 3.3. As we can see there is no net force acting on the cylinder, the pressure on the upper half of the cylinder is the same as the pressure on the lower half, so no lift is achieved.

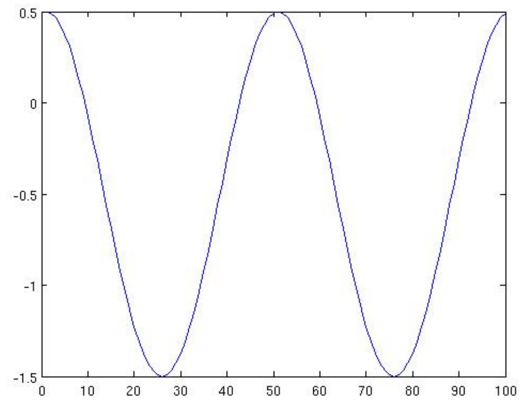


Figure 3.3: Pressure on the circle (no circulation)

3.1.2 Super positioning

There is another way of determining the solution of this problem, which gives an exact solution. I have used this method as a check for my results obtained with the BIM. This time we are going to use singularities. These are functions that can be combined to construct flow fields. The most familiar singularities are the source, doublet and vortex. In some cases, like this one, the singularities can be placed inside the body. But to construct flows around arbitrary shape this cannot be done. For this we need to put singularities on the boundary in such a way that we get the right flow field . We will use this later on.

From [8] we know that the solution to this flow is given by super positioning a uniformal parallel flow $(U, 0)$ with a doublet in the origin. De complex speed potentials of these flows are given by $\chi_1(z) = -Uz$ (it's a flow from right to left) and $\chi_2(z) = -\frac{M}{2\pi z}$ respectively, where M is the doublet moment. Since we are looking for the flow around a cylinder with radius a we take $M = 2\pi U a^2$. So we get:

$$\chi(z) = \chi_1 + \chi_2 = -U(z + \frac{a^2}{z})$$

If we now look at the complex speed $\omega(z)$ we see that:

$$\omega(z) = \chi'(z) = -U(1 - \frac{a^2}{z^2})$$

As we saw before the streamlines are given by $\Psi(z) = c$:

$$\Im(\chi(z)) = \Psi = -U(y - \frac{a^2 y}{x^2 + y^2}) = c$$

We take the constant to be 0. Now $\Psi(z) = 0$ is given by $y = 0$ and the circle $x^2 + y^2 = a^2$ and so we indeed found the flow around a circle with radius a. We also see that there are two stagnation points, namely at $z = \pm a$, because $\omega(\pm a) = 0$. The potential Φ is given by:

$$\Re(\chi(z)) = \Phi = -U(x + \frac{a^2 x}{x^2 + y^2})$$

Since $U = 1$ and we are looking at Φ on the surface $x^2 + y^2 = a^2$ we have $\Phi = -2a\cos(\theta)$. We can determine the velocity and pressure in the same way as we did with the BIM.

3.1.3 Kutta-Joukowski

The Kutta-Joukowski theorem states a relationship between the circulation around an airfoil and the lifting force acting upon it. The relation is as following:

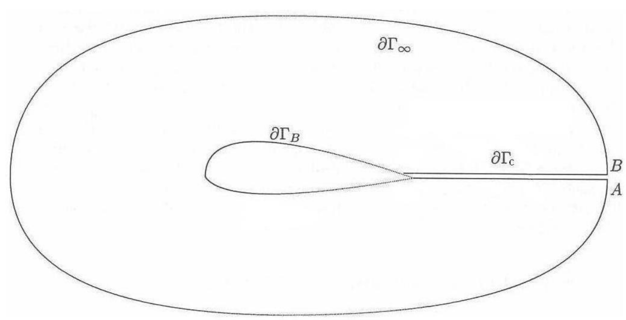
$$L = \rho V \Gamma$$

In which L is the force experienced in the direction normal to the velocity V, ρ is the air density and Γ is the rotation around the airfoil. So this fact states that if there is no circulation around the airfoil we cannot have lift. This theorem is essential for our further calculations.

Another important fact we need is the Kutta-Joukowski condition. This states that when dealing with airfoils the flow should leave the body smoothly at the trailing end. This is simply a condition such that the flow doesn't go around sharp edge at the trailing end, which is simply not possible in reality. For more information I refer to [5, 7].

3.2 Flow around a cylinder with circulation

When dealing with flows with circulation we come across a special problem. Also see [3, 6]. We stated in the last section that if we have no circulation, we cannot have lift. Lets take a look at the circulation around the profile shown in the figure below.



The circulation at the boundary at "infinity" (denoted in the figure as $\delta\Gamma_\infty$) is given by:

$$C_{\Gamma_\infty} = \int_A^B v \cdot e_s ds = \int_A^B \frac{\delta\Phi}{\delta s} ds = \Phi_B - \Phi_A$$

Here we see the problem. When we assume Φ is continuous, we see that:

$$\lim_{A \to B} \Gamma = \lim_{A \to B} (\Phi_B - \Phi_A) = 0$$

So there is no circulation and as a consequence there cannot be any lift! So the only thing we can do is to make sure that Φ is not continuous. We do this by making a slit in the domain, from the profile to infinity and let the potential make a jump over this cut (η_c). So in our results we should see a discontinuity in Φ.

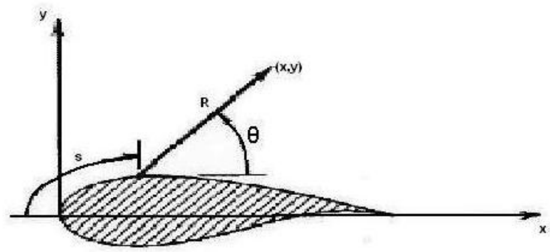
3.2.1 Boundary integral method

For this problem we are also going to use a BIM. This method will be a little different from the one we used to calculate the potential on the cylinder without circulation. Again we assume that the velocity at infinity is given by $V_\infty = (1, 0)$.

Just as in the case of no circulation, first we are going to derive a suitable form for the Laplace equation. For this we are going to use singularities. As said before, we can construct arbitrary flows by putting singularities on the boundary. We are going to represent the potential by combining sources and vortices on the boundary. So

$$\Phi(x, y) = \Phi_\infty(x, y) + \Phi_S(x, y) + \Phi_V(x, y)$$

Here Φ_S and Φ_V are the potentials due to the sources and vortices respectively.



They are given by $\Phi_S = \int_S [\frac{q(s)}{2\pi} \ln(R)]dS$ and $\Phi_V = -\int_S [\frac{\gamma(s)}{2\pi} \cdot (\theta)]dS$, where R and θ are defined as in the figure. So now we get:

$$\Phi(x, y) = \Phi_\infty(x, y) + \int_S \left[\frac{q(s)}{2\pi} \ln(R) - \frac{\gamma(s)}{2\pi} \cdot (\theta)\right] dS$$

Here $q(s)$ and $\gamma(s)$ are the source and vortex strength in the point s. So the problem is to find the strength of the sources and vortices. We use Cartesian coordinates because this is a method for arbitrary flows, so its not always an advantage to change to polar coordinates.

3.2.2 Hess and Smith method

Now for the panel method. The method we are going to discuss here is the classic Hess and Smith method for arbitrary shapes, as found in [4]. In this approach we first divide the surface into straight line segments. We assume that the source strength is constant over each panel, but has a different value for each panel, and the vortex strength is constant and equal over each panel. Again we are going to use the same panels as in the case with no circulation. So again we have N panels on which are N nodes in which we are going to determine the velocity. We can rewrite the integral for this case to

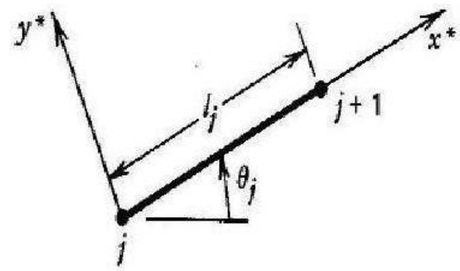
$$\Phi(x, y) = V_\infty x + \sum_j \int_{\Gamma_j} \left[\frac{q(s)}{2\pi} \ln(r) - \frac{\gamma}{2\pi} \cdot \theta\right] d\Gamma$$

Here Γ_j is the length of je jth panel. Since we assumed that $q(s)$ is constant over each panel, we can write $q(s) = q_i \; i = 1, .., N$. So we need to find N values of q_i and only one value of γ.

Define Θ_i to be the inclination of panel i with respect to the x axis. If we define l_i to be the length of the i^{th} panel we can say that

$$\sin(\Theta_i) = \frac{y_{i+1} - y_i}{l_i} \qquad \cos(\Theta_i) = \frac{x_{i+1} - x_i}{l_i}$$

The normal (outwards) and tangential vectors are given by



$$\mathrm{n}_i = (\sin(\Theta_i),\ -\cos(\Theta_i))$$

$$\mathrm{t}_i = (\cos(\Theta_i),\ \sin(\Theta_i))$$

We will now try to compute the unknowns by using the boundary conditions plus the Kutta condition. The boundary conditions say that there is no velocity in the normal direction, so $V_i \cdot n_i = 0$, or

$$u_i \sin(\Theta_i) - v_i \cos(\Theta_i) = 0 \qquad \text{for each } i = 1...N$$

The last equation is found by using the Kutta condition. In practice this condition states that on the trailing edge the pressure on the lower surface is equal to the pressure on the upper surface, so the flow leaves smoothly at the trailing end. Here we satisfy the condition by stating that the tangential velocity on the N^{th} panel is equal to that on the first panel, but in opposite direction. If we write this down we get

$$V_1 \cdot t_1 = -V_N \cdot t_N$$

$$u_1 \cos(\Theta_1) + v_1 \sin(\Theta_1) = -u_N \cos(\Theta_N) - v_N \sin(\Theta_N)$$

In order to solve this system of equations we are going to do the following:

1. Write down the velocities, u_i and v_i in terms of contributions from all the singularities.
2. Find the algebraic equations defining the influence coefficients.
3. Use the boundary conditions to get N equations for the q_i and γ
4. Write down the Kutta equation for the $N + 1^{th}$ equation
5. Solve the system for the q_i and γ , using FFT's and computing the tangential velocity V_t and the pressure p

Step 1: Velocities The velocities at any point are given by contributions from the velocities induced by the source and vortex distributions over each panel. We can write this as:

$$u_i = V_\infty + \sum_{j=1}^{N} q(j) u_{s_{ij}} + \gamma \sum_{j=1}^{N} u_{v_{ij}}$$

$$v_i = \sum_{j=1}^{N} q(j) v_{s_{ij}} + \gamma \sum_{j=1}^{N} v_{v_{ij}}$$

Where $u_{s_{ij}}, u_{v_{ij}}, v_{s_{ij}}$ and $v_{v_{ij}}$ are the influence coefficients. For example $u_{s_{ij}}$ is the x-component of the velocity at x_i due to a unit source distribution on the panel j.

Step 2: Determine the influence coefficients To find an algebraic expression for the influence coefficients we need to change to another coordinate system which we will call (x^*, y^*). We will locally align this system with each panel, as seen in the figure. We can switch between this local coordinate system and the global coordinate system by

$$u = u^* \cos(\Theta_j) - v^* \sin(\Theta_j)$$

$$v = u^* \sin(\Theta_j) + v^* \cos(\Theta_j)$$

Now we can calculate the velocities induced by the singularities. First the source distribution. The velocity field induced by a source in Cartesian coordinates is given by:

$$u(x,y) = \frac{Q}{2\pi} \frac{x}{x^2 + y^2}, \qquad v(x,y) = \frac{Q}{2\pi} \frac{y}{x^2 + y^2}$$

If we want to know the velocity induced by sources located along the x-axis at a point t until a point l, we can say that

$$u_s = \int_0^l \frac{q(t)}{2\pi} \frac{x - t}{(x-t)^2 + y^2} dt$$

$$v_s = \int_0^l \frac{q(t)}{2\pi} \frac{y}{(x-t)^2 + y^2} dt$$

If we now use our local coordinate system on panel j, and say $q(t) = 1$ we get

$$u^*_{s_{ij}} = \frac{1}{2\pi} \int_0^{l_j} \frac{x_i^* - t}{(x_i^* - t)^2 + y_i^{*2}} dt$$

$$v^*_{s_{ij}} = \frac{1}{2\pi} \int_0^{l_j} \frac{y_i^*}{(x_i^* - t)^2 + y_i^{*2}} dt$$

These integrals are found to be

$$u^*_{s_{ij}} = \frac{1}{2\pi} \left[\ln \left[(x_i^* - t)^2 + y_i^{*2} \right]^{\frac{1}{2}} \right]_0^{l_j}$$

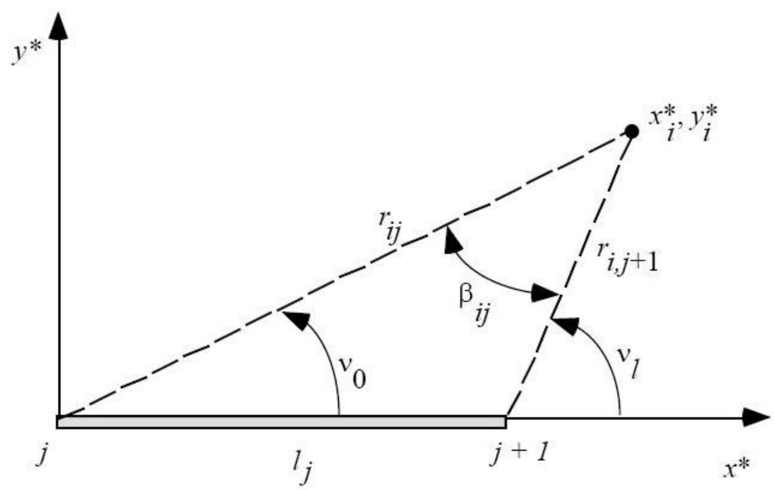
$$v^*_{s_{ij}} = \frac{1}{2\pi} \left[\tan^{-1} \left(\frac{y_i^*}{x_i^* - t} \right) \right]_0^{l_j}$$

We will now examine what this expression means. If we evaluate these expressions we can say that

$$u^*_{s_{ij}} = -\frac{1}{2\pi}\ln\left[\frac{r_{i,j+1}}{r_{i,j}}\right]$$

$$v^*_{s_{ij}} = \frac{\beta_{ij}}{2\pi}$$

Here r_{ij} is the distance between the node i and the begin point of panel j. The angle β_{ij} is the angle at which $r_{i,j}$ and $r_{i,j+1}$ stand.



If we now look at the case in which $i = j$ we see that $u^*_{s_{ii}} = 0$ and β_{ii} depends on the side from which we approach the panel. If we approach from the inside $\beta_{ii} = -\pi$ and if we approach from the outside $\beta_{ii} = \pi$. But since we are working on the outside problem we get $\beta_{ii} = \pi$.

In the same way we can the influence coefficients from the vortices. The final result yields

$$u^*_{v_{ij}} = \frac{\beta_{ij}}{2\pi}$$

$$v^*_{v_{ij}} = \frac{1}{2\pi}\ln\left[\frac{r_{i,j+1}}{r_{i,j}}\right]$$

Step 3: Finding the N equations Now we are trying to find a system of equations that has the following form

$$\sum_{j=1}^{N} A_{ij}q_j + A_{i,N+1}\gamma = b_i \qquad i = 1, ..., N$$

Our boundary condition is given by

$$u_i \sin(\phi_i) - v_i \cos(\phi_i) = 0 \qquad \text{for each panel } i$$

We also know that the velocities are given by

$$u_i = V_\infty + \sum_{j=1}^{N} q_j u_{s_{ij}} + \gamma \sum_{j=1}^{N} u_{v_{ij}}$$

$$v_i = \sum_{j=1}^{N} q_j v_{s_{ij}} + \gamma \sum_{j=1}^{N} v_{v_{ij}}$$

If we substitute this, we get

$$\overbrace{V_\infty \sin(\Theta_i)}^{-b_i} + \sum_{j=1}^{N} \overbrace{\left(\sin(\Theta_i) u_{s_{ij}} - \cos(\Theta_i) v_{s_{ij}}\right)}^{A_{ij}} q_j$$

$$+ \overbrace{\left(\sin(\Theta_i) \sum_{j=1}^{N} u_{v_{ij}} + \sin(\Theta_i) \sum_{j=1}^{N} v_{v_{ij}}\right)}^{A_{i,N+1}} \gamma = 0$$

We can now calculate this by using the fact that

$$u_{s_{ij}} = u^*_{s_{ij}} \cos(\Theta_j) - v^*_{s_{ij}} \sin(\Theta_j)$$

$$v_{v_{ij}} = u^*_{v_{ij}} \sin(\Theta_j) + v^*_{v_{ij}} \cos(\Theta_j)$$

and substitute this into the equation. Using some trigonometric identities we finally obtain:

$$A_{ij} = -\frac{1}{2\pi} \sin(\Theta_i - \Theta_j) \ln\left(\frac{r_{i,j+1}}{r_{i,j}}\right) - \frac{1}{2\pi} \cos(\Theta_i - \Theta_j)\beta_{ij}$$

$$A_{i,N+1} = \frac{1}{2\pi} \sum_{j=1}^{N} \left(\cos(\Theta_i - \Theta_j) \ln\left(\frac{r_{i,j+1}}{r_{i,j}}\right) - \sin(\Theta_i - \Theta_j)\beta_{ij}\right)$$

$$b_i = V_\infty \sin(\Theta_i)$$

Step 4: Obtain equation N+1 So far we have got $N+1$ unknowns and N equations. This means we have to have another equation for a unique solution. For this we use the Kutta condition, given by

$$u_1 \cos(\Theta_1) + v_1 \sin(\Theta_1) = -u_N \cos(\Theta_N) - v_N \sin(\Theta_N)$$

Again we are substituting this into the velocities given in step 1

$$u_1 = V_\infty + \sum_{j=1}^{N} q_j u_{s_{1j}} + \gamma \sum_{j=1}^{N} u_{v_{1j}}$$

$$v_1 = \sum_{j=1}^{N} q_j v_{s_{1j}} + \gamma \sum_{j=1}^{N} v_{v_{1j}}$$

$$u_N = V_\infty \cos(\alpha) + \sum_{j=1}^{N} q_j u_{s_{Nj}} + \gamma \sum_{j=1}^{N} u_{v_{Nj}}$$

$$v_N = \sum_{j=1}^{N} q_j v_{s_{Nj}} + \gamma \sum_{j=1}^{N} v_{v_{Nj}}$$

We are trying to get an equation of the form

$$\sum_{j=1}^{N} A_{N+1,j} q_j + A_{N+1,N+1}\gamma = b_{N+1}$$

So if we substitute the velocity equations into the Kutta condition and regroup this we get

$$\overbrace{V_\infty(\cos(\Theta_1) + \cos(\Theta_N))}^{-b_{N+1}}$$

$$+\overbrace{\sum_{j=1}^{N}\Big(\cos(\Theta_1)u_{s_{1j}} + \sin(\Theta_1)v_{s_{1j}} + \cos(\Theta_N)u_{s_{Nj}} + \sin(\Theta_N)v_{s_{Nj}}\Big)}^{A_{N+1,j}} q_j$$

$$+\overbrace{\sum_{j=1}^{N}\left(\cos(\Theta_1)\sum_{j=1}^{N} u_{v_{1j}} + \sin(\Theta_1)v_{v_{1j}} + \cos(\Theta_N)\sum_{j=1}^{N} u_{v_{Nj}} + \sin(\Theta_N)v_{v_{Nj}}\right)}^{A_{N+1,N+1}} \gamma = 0$$

again we can simplify this towards the final form:

$$A_{N+1,j} = \frac{1}{2\pi}\,[\sin(\Theta_1 - \Theta_j)\beta_{1j} + \sin(\Theta_N - \Theta_j)\beta_{Nj}$$

$$-\cos(\Theta_1 - \Theta_j)\ln\left(\frac{r_{1,j+1}}{r_{1,j}}\right) - \cos(\Theta_N - \Theta_j)\ln\left(\frac{r_{N,j+1}}{r_{N,j}}\right)\Bigg]$$

$$A_{N+1,N+1} = \frac{1}{2\pi}\sum_{j=1}^{N}\left[\sin(\Theta_1 - \Theta_j)\ln\left(\frac{r_{1,j+1}}{r_{1,j}}\right) + \sin(\Theta_N - \Theta_j)\ln\left(\frac{r_{N,j+1}}{r_{N,j}}\right)\right.$$

$$\left. + \cos(\Theta_1 - \Theta_j)\beta_{1j} - \cos(\Theta_N - \Theta_j)\beta_{Nj}\right]$$

$$b_{N+1} = -V_\infty(\cos(\Theta_1) + \cos(\Theta_N))$$

Step 5: Solving the system Now we have found the $N+1$ equations we can solve the $N+1$ unknowns. We have a system of the following form:

$$Ax = b$$

Here $x = (q_1, q_2, ..., q_N, \gamma)$. This system can be solved using FFT's. For this we need to decompose A and b into the following form

$$A = \begin{pmatrix} C & c \\ d & e \end{pmatrix}$$

$$b = \begin{pmatrix} f \\ g \end{pmatrix}$$

with C an $N \times N$ ciruclant matrix, c and f vectors of length N, d a flat vector of length N and e and g scalars. If we now pre multiply the first N rows with C^{-1} we get

$$\begin{pmatrix} I & C^{-1}c \\ d & e \end{pmatrix} \cdot x = \begin{pmatrix} C^{-1}f \\ g \end{pmatrix}$$

We can now let $d = 0$, with the help of some row transformations. We get

$$\begin{pmatrix} I & C^{-1}c \\ 0 & e - dC^{-1}c \end{pmatrix} \cdot x = \begin{pmatrix} C^{-1}f \\ g - C^{-1}f \end{pmatrix}$$

Since $e - dC^{-1}c$ and $g - C^{-1}f$ are scalars, we can just write down $x_{N+1} = \gamma$. So for $i = 1, \ldots, N$ we hold

$$x_i = q_i = C^{-1}b - \gamma \cdot C^{-1}f$$

This system can be solved using FFT's.

In the case that we work with we actually get some more simplification. If we display A (with $N = 9$ and $\Gamma = -\pi$) we see that c only consists of zeros.

$$A = \begin{pmatrix}
-0.5 & 0.050 & 0.056 & 0.057 & 0.057 & 0.057 & 0.057 & 0.056 & 0.050 & 0 \\
0.050 & -0.5 & 0.050 & 0.056 & 0.057 & 0.057 & 0.057 & 0.057 & 0.056 & 0 \\
0.056 & 0.050 & -0.5 & 0.050 & 0.056 & 0.057 & 0.057 & 0.057 & 0.057 & 0 \\
0.057 & 0.056 & 0.050 & -0.5 & 0.050 & 0.056 & 0.057 & 0.057 & 0.057 & 0 \\
0.057 & 0.057 & 0.056 & 0.050 & -0.5 & 0.050 & 0.056 & 0.057 & 0.057 & 0 \\
0.057 & 0.057 & 0.057 & 0.056 & 0.050 & -0.5 & 0.050 & 0.056 & 0.057 & 0 \\
0.057 & 0.057 & 0.057 & 0.057 & 0.056 & 0.050 & -0.5 & 0.050 & 0.056 & 0 \\
0.056 & 0.057 & 0.057 & 0.057 & 0.057 & 0.056 & 0.050 & -0.5 & 0.050 & 0 \\
0.050 & 0.056 & 0.057 & 0.057 & 0.057 & 0.057 & 0.056 & 0.050 & -0.5 & 0 \\
-1.102 & -1.547 & -0.658 & -0.278 & 0 & 0.278 & 0.658 & 1.547 & 1.101 & 0.120
\end{pmatrix}$$

So now we get

$$\begin{pmatrix} I & 0 \\ 0 & e \end{pmatrix} \cdot x = \begin{pmatrix} C^{-1}f \\ g - C^{-1}f \end{pmatrix}$$

Since A now has a diagonal form we can just compute x using just N calculations.

The fact that c only consists of zeros is due to the fact that the source strenght is not dependant on the circulation. This is because the sources replicate the shape of the cylinder and the circulation creates the flow around it.

Now that we know all the q_i and γ, we can calculate the tangential velocity in the nodes.

$$V_{t_i} = u_i \cdot \cos(\Theta_i) + v_i \cdot \sin(\Theta_i)$$

with as before

$$u_i = V_\infty + \sum_{j=1}^{N} q_j u_{s_{ij}} + \gamma \sum_{j=1}^{N} u_{v_{ij}}$$

$$v_i = \sum_{j=1}^{N} q_j v_{s_{ij}} + \gamma \sum_{j=1}^{N} v_{v_{ij}}$$

The pressure is then determined in the same way as before, using the Bernoulli law.

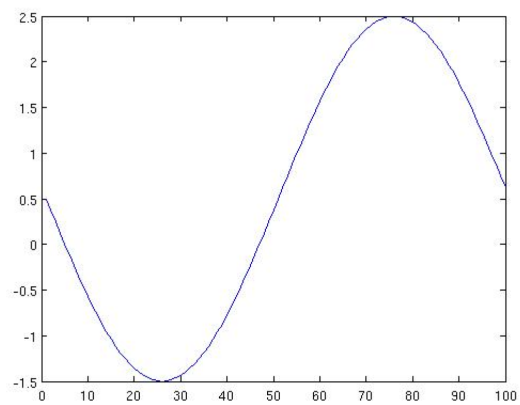


Figure 3.4: Tangential velocity on the circle (with circulation $\Gamma = -2\pi$)

3.2.3 Super positioning

Just as in the case of a flow with no circulation we will now solve the problem using super positioning. The complex speed potential of a circulation in the origin is given by $\chi(z) = -\frac{i\Gamma}{2\pi}\ln(z)$ with $\Gamma \leq 0$ (counter clock wise). So our new complex speed potential is:

$$\chi(z) = -U(z + \frac{a^2}{z}) - \frac{i\Gamma}{2\pi}\ln(z)$$

Before we examine this flow we will introduce polar coordinates, $z = re^{i\theta}$. Now

$$-U(re^{i\theta} + \frac{a^2}{r}e^{-i\theta}) - \frac{i\Gamma}{2\pi}(\ln(r) + i\theta)$$

and thus:

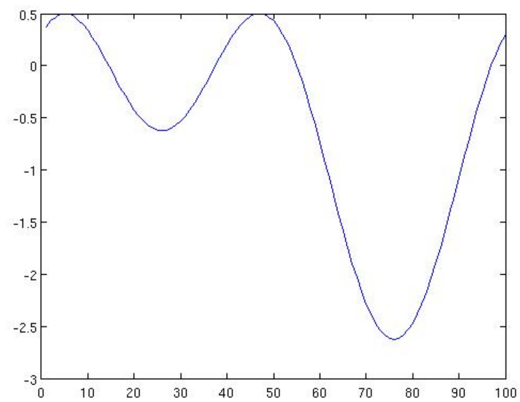


Figure 3.5: pressure on the circle (with circulation $\Gamma = -2\pi$)

$$\Phi(r,\theta) = -U(r + \frac{a^2}{r^2})\cos(\theta) + \frac{\Gamma\theta}{2\pi})$$

We can now calculate the radial speed and the tangential speed:

$$v_{rad}(r,\theta) = \Phi_r(r,\theta) = -U(1 - \frac{a^2}{r^2})\cos(\theta)$$

$$v_{\tan}(r,\theta) = \frac{1}{r}\Phi_\theta(r,\theta) = U(1 + \frac{a^2}{r^2})\sin(\theta) + \frac{\Gamma}{2\pi r}$$

Since $v_{rad} = 0$ for all points on the circle we still have a flow around a cylinder. But if we look at the stagnation points we see that they have shifted. For $|\Gamma| < 4\pi U a$ we have that these points have shifted to the places were $\sin(\theta) = \frac{\Gamma}{4\pi U a}$. So by adding a circulation we can determine were the flow has its stagnation points. We will need this when we examine the flow around an better airfoil model.

As we see in 3.5 we can see that the pressure is not equal on both the upper side and the lower side of the cylinder anymore. The pressure on the upper half is much lower that on the lower half, hence there is a lift force acting on the cylinder.

Chapter 4

Conformal mappings

A conformal mapping is an angle preserving coordinate transformation which projects the complex plane $\zeta = \nu + i \cdot \eta$ onto another complex plane $z = x + i \cdot y$ with the help of a function f: $z = f(\zeta)$. Conformal mapping have the property that they transform potential equations into other potential equations. The most renown conformal mapping is the Joukowski transformation.

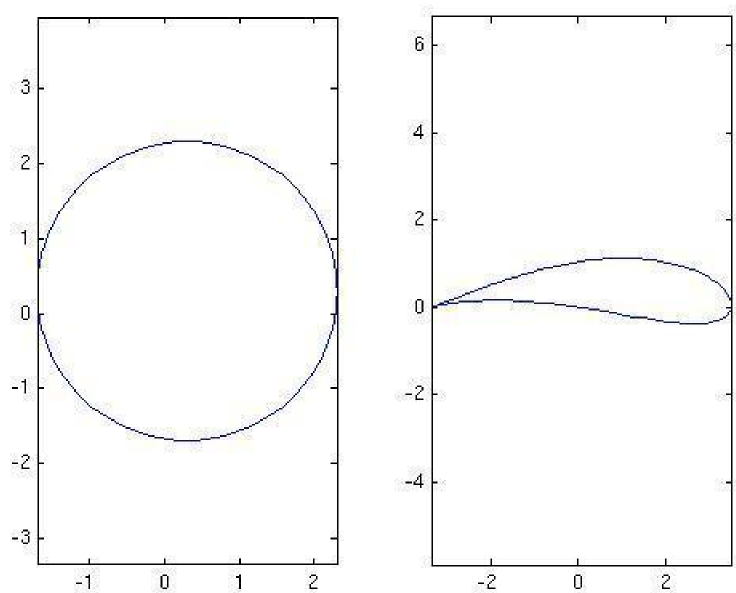


Figure 4.1: Joukowski transformation

Joukowski transformation

The Joukowski-mapping is given by

$$z = \zeta + \frac{\lambda^2}{\zeta}$$

If we take $\lambda = a$, then this mapping maps circles in the ζ-plane with a radius a onto the segment $[-2a, 2a]$ in the z-plane. We can see this in the following way. A circle with radius a in the ζ-plane is given by $\zeta = ae^{i\theta}$with $0 \leq \theta \leq 2\pi$. If we now map it onto the z-plane using the Joukowski mapping we get: $z = ae^{i\theta} + \frac{a^2}{ae^{i\theta}} = a(e^{i\theta} + e^{-i\theta}) = 2a\cos(\theta)$ with $0 \leq \theta \leq 2\pi$. This is the same as a flat plate with length $4a$. So if we use the Joukowski mapping on the speed field of the flow around a cylinder we would get the flow around a flat plate.

Since a flat plate and a circle aren't good models for an airfoil we will need to take a better look at the Joukowski mapping. Let us take a circle with radius a in the $\zeta-$plane that doesn't have its centre in the origin $(\nu, \eta) = (0, 0)$. Say we place it at $(\nu, \eta) = (\alpha, \beta)$. Now let $\lambda = -\alpha + \sqrt{r^2 - \beta^2}$. If we now use the Joukowski mapping we can see in 4.1 that the circle is transformed in an airfoil shape.

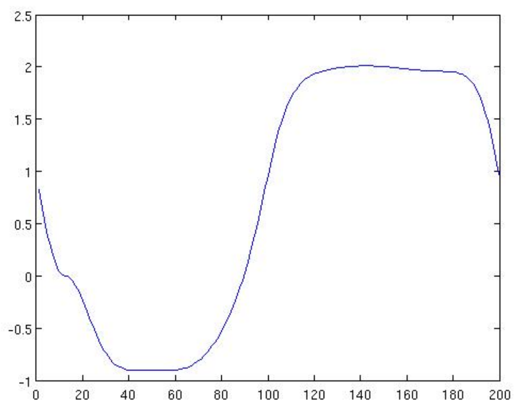


Figure 4.2: Tangential velocity on the Joukowski airfoil

In order to satisfy the Kutta condition we must have that the flow leaves the trailing edge of the airfoil smoothly. To do this we must shift the stagnation point to the trailing edge. As we saw we can do this by adding circulation to the flow. So the problem is to find how large this circulation has to be to satisfy the Kutta condition. From [2] we know that the stagnation points are given by θ for which

$$\sin(\theta) = \frac{-\Gamma}{4\pi a}$$

$$\Gamma = -4\pi a \sin(\theta)$$

So now if we know θ, and thus we know Γ.

We need the circulation to leave the cylinder at the point that transforms to the trailing edge. If we take another look at the transformation we see that the points on the cylinder for which $x = 0$ are transformed to the leading and trailing edge of the airfoil. In our case the trailing edge is formed by the left point for which $x = 0$. Now we can compute θ since we know (ν, η), the centre of the cylinder. And since we know θ we can compute Γ.

For example, let's take the centre of the circle with radius 2 to be at $(\nu, \eta) = (0.3, 0.3)$. So the angle $\alpha = \sin^{-1}(\frac{\eta}{2}) = 0.15$ radians and $\theta = \pi + \alpha = 3,29$ radians. So we need to have a circulation of $\Gamma = -4\pi a \sin(\theta) = 3,77$.

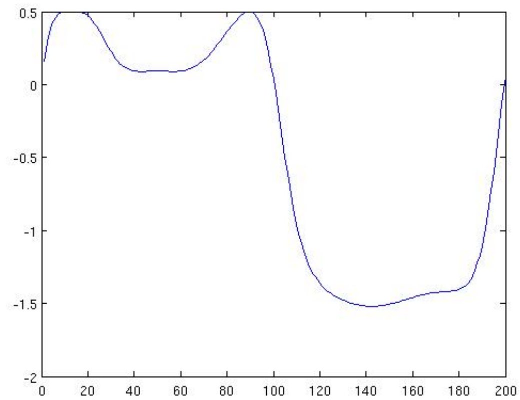


Figure 4.3: Pressure on the the Joukowski airfoil

Now we know how large Γ is we can transform the speed V in the ζ plane to the complex speed W in the z plane. Recall that the complex speed W in the z plane is given by

$$W = \frac{d\chi}{dz} = \frac{d\chi}{d\zeta} \cdot (\frac{dz}{d\zeta})^{-1} = V \cdot \frac{\zeta^2}{\zeta^2 - \lambda^2}$$

With this complex speed W we can calculate the tangential velocity W_t on the panels. This is shown in 4.2.

The last thing we need to know is determining the pressure on the airfoil. This is again done Bernoulli's law and the results are displayed in 4.3.

Chapter 5

Discussion

Now we have succesfully transformed the flow onto a Joukowski airfoil we will take a closer look at some of the most interesting results. First of we are going to take a look at the potential on the cylinder with circulation. As seen in 5.1 we see that indeed the potential makes a jump, just as we said in the beginning of section 3.2. So the conditions for lift are met.

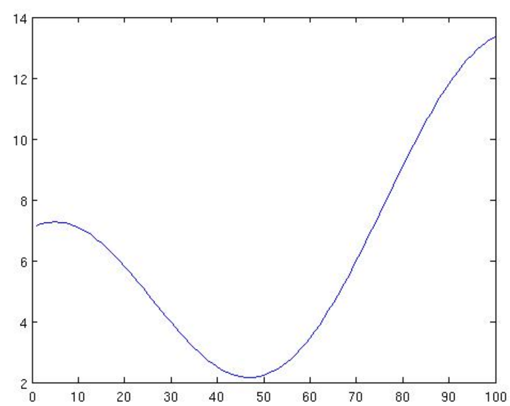


Figure 5.1: Potential with circulation

Second we take a look at the pressure on the Joukowski wing (4.3). We should be careful when we look at this figure. This is because the Joukowski transformation doesn't transform the panels on panels with equal length. The panels close to the trailing edge and close to the tip are very small. This means that the pressure on these panels doesn't have a great influence on the upward force.

If we take a closer look at the horizontal speed on the airfoil, shown in 5.2, we see that the speed in the first few nodes is positive. This is not what one would expect since it the flow should everywhere be going in the opposite direction. So why is it positive?

We can explain this by looking closer at the Joukowski transformation. Actually the transformation does not have a sharp edge at the left side. In fact the Joukowski mapping transforms the circle in a ∞-shape. But the left loop is so small that it looks as if the shape has a sharp edge. So that is the reason that in the first few point the horizontal speed is positive.

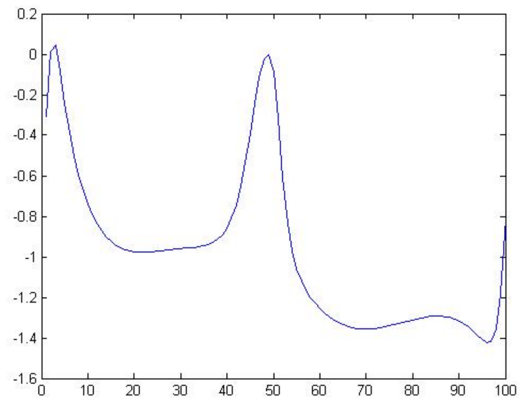


Figure 5.2: Horizontal speed around the airfoil

In conclusion: we have found a method that can transform the flow around a cilinder onto a Joukowski airfoil, so our goal has been reached. The only thing left to say is that the results found by this method are limited in their usefulness. The fact that we have used potential theory means that we ignore rotations in the flow. Offcourse in reallity these rotations are present, especially at high speeds. Another thing is: when moving at high speeds the flow doesn't follow the profile anymore, but separates from the wing. So these results are quite reliable, but only at low speeds, as seen when the airplane is lifting off.

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