Controlling the Added-mass Instability in Fluid-solid Coupling

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1 Introduction

Simulating the hydrodynamics of floating structures using a partitioned strategy poses a major challenge when the coupling between the fluid and the structure is strong. The incompressibility of the fluid plays an important role, and leads to strong coupling when the ratio of so-called added mass to structural mass is considerate. Existing fluid-structure interaction procedures become less efficient in such cases, and can even become unstable as shown by Causin et al., 2004. The current paper proposes a coupling method that is unaffected by this instability, and shows its efficient implementation.

2 Model

The fluid model and the solid model should be in agreement at any point where the fluid meets the solid. Restricting ourselves to mechanical interactions, this agreement requires two conditions to be satisfied at each point. The kinematic condition simply requires that the local fluid and solid motions are the same (i.e. displacement, velocity, acceleration), so each point remains on the interface. The dynamic condition requires that any force on the fluid is equalled by an opposite force on the solid in order to conserve momentum.

Typically these two conditions are enforced by a fixed-point iteration. The fluid and solid models are evaluated in turn, subject to the kinematic and dynamic condition respectively. Assume the interface is discretized into n degrees of freedom. The displacements $\mathbf{x}(t) \in \mathbb{R} \to \mathbb{R}^n$ and forces $\mathbf{f}(t) \in \mathbb{R} \to \mathbb{R}^n$ are exchanged between the fluid and the solid. Everything else remaining the same, the stability of this iteration can therefore be expressed in terms of the response of each model to the applied forces and displacements. We denote the fluid response by the function F and the solid response by the function S,

$$\mathbf{f}(t) = F(\ddot{\mathbf{x}}(t)), \tag{1a}$$

$$\ddot{\mathbf{x}}(t) = S(\mathbf{f}(t)). \tag{1b}$$

Of course these functions depend on the initial state and external forcing of the models as well, but this is understood to be part of F and S since these do not change between iterations. The solid only responds to the fluid force, and the fluid in turn responds to the solid motion. The fixed-point iteration can thus be expressed as

$$\mathbf{f}_{k+1}(t) = F(S(\mathbf{f}_k(t))). \tag{2}$$

The asymptotic convergence is determined by the spectral radius of the Jacobian of $F \circ S$ at the fixed point,

$$J(t) = \frac{\partial F \circ S(\mathbf{f}(t))}{\partial \mathbf{f}(t)} \tag{3}$$

which is a function of time. Only if the the spectral radius is smaller than one, the above fixed-point iteration (2) will converge. Otherwise, one has to resort to relaxation or quasi-Newton methods.

When timestepping an unsteady fluid-solid problem (1), the functions $\mathbf{f}(t)$ and $\ddot{\mathbf{x}}(t)$ are converged up to the start t_n of the timestep and are to be determined up to the end of the timestep $t_n + \Delta t$ via a suitable extrapolation. In an incompressible flow, the only nonzero eigenvalues of $J(t_n)$ are related to the inertia of the fluid and solid (Seubers and Veldman, 2017). The ratio of added mass of the fluid to the the solid mass determines these eigenvalues (Förster et al., 2006) and hence the stability. This instability is not controllable by the timestep Δt.

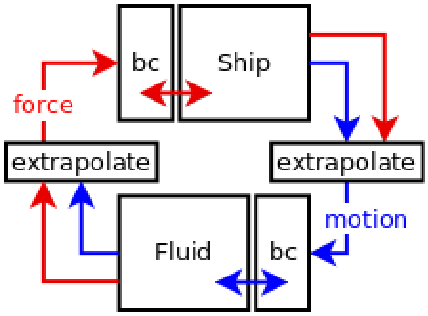

Fig. 1: Fixed point iteration according to Eq. (2).

3 Method

It is undesirable that the stability of the numerical process depends on a physical parameter that can not be controlled. Therefore we will introduce a modification to the fixed-point iteration (2) that removes the added-mass instability from the system (1). In that system, the fluid is directly subjected to the motion S of the solid. This boundary condition is only realistic when the solid is largely unaffected by the fluid. Instead, we can subject the fluid to a more realistic condition that allows the solid to respond.

The key observation is that the two conditions (kinematic and dynamic) on the two unknowns ($\ddot{\mathbf{x}}$ and $\mathbf{f}$) can be linearly combined. However we cannot simply combine (1a) with (1b), since this would require a single monolithic code for the fluid and solid. Instead, we can make a crude approximation of the solid,

$$\ddot{\mathbf{x}} = S(\mathbf{f}) \approx S(\mathbf{f}_k) + D(\mathbf{f} - \mathbf{f}_k). \tag{4}$$

where the n-by-n matrix D contains the crude dynamics. Instead of the pure kinematic condition, this equation becomes the new boundary condition to the fluid. Substituting Eq. (4) into Eq. (1a) produces a modified iteration process

$$\mathbf{f}_{k+1}(t) = F\Big(S(\mathbf{f}_k(t)) + D(\mathbf{f}_{k+1}(t) - \mathbf{f}_k(t))\Big). \tag{5}$$

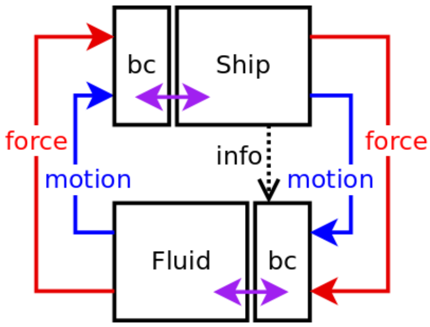

Fig. 2: Fixed point iteration according to Eq. (5).

We wish to point out three properties of the modified iteration (5). First of all, it has exactly the same solutions as Eq. (2) since these equations are identical at the fixed point $\mathbf{f}_{k+1} = \mathbf{f}_k$. This is due to the fact that although (4) can be a crude approximation, it is equal by definition to the exact solid at these fixed points. Secondly, the stability of the fixed points will be different. Comparing the Jacobians of the original (2) and modified (5) iterations, we see that

$$J_{\text{orig}}(t) = J_F(t) * J_S(t), \tag{6}$$

$$J_{\text{mod}}(t) = (J_F^{-1}(t) - D)^{-1} * (J_S(t) - D), \tag{7}$$

hence the matrix D can be used to control the stability. Thirdly, if the matrix D equals the linearized solid J_S the method will convergence with second order in the nonlinearity, comparable to Newtons method.

4 Control of stability

To construct an appropriate dynamic matrix D, we need to assume some knowledge about the solid and fluid response functions. The more specific this knowledge, the more we can reduce the convergence factor. On the other hand, we may restrict the assumptions to a minimum and look for the most general matrix D that will guarantee stability.

We start with a minimum of two assumptions: the Jacobian of the solid J_S is symmetric positive semidefinite, and the Jacobian of the fluid J_F is symmatric negative definite. Informally, this means that the solid motion changes in the direction of the forcing, and the fluid resists this motion by changing the force in the opposite sense. This allows us to bound the spectral radius r of the modified iteration,

$$r(J_{\text{mod}}) \leq r((J_F^{-1} - D)^{-1}) r(J_S - D). \tag{8}$$

Now consider the most simple approximation of the dynamics $D = \alpha I$. Since J_F is negative definite, the shift will make it more negative $r((J_F^{-1} - \alpha I)^{-1}) < \alpha^{-1}$. On the other hand, there is a bound due to the approximation of the solid $r(J_S - \alpha I) = \max(\alpha, r(J_S) - \alpha)$.

$$r(J_{\text{mod}}) < \alpha^{-1} \max(\alpha, r(J_S) - \alpha) \tag{9}$$

Stability is guaranteed if we choose $\alpha > \frac{r(J_S)}{2}$. Of course, convergence can be improved with more specific approximations of the solid dynamics. Another stable approximation is obtained when using the exact mass properties of the solid, ignoring any stiffness or damping effects.

5 Implementation of the boundary condition

As noted in section 3, the boundary conditions to the fluid need to be modified. The combined kinematic and dynamic condition (4) is expressed as a linear combination of forces and motions. Consider an incompressible viscous fluid governed by the Navier-Stokes equations,

$$\frac{\partial \vec{u}}{\partial t} + \vec{u}^T \nabla \vec{u} + \frac{1}{\rho} \nabla p = \nu \nabla^2 \vec{u} + \vec{g}, \qquad \nabla^T \vec{u} = 0. \tag{10}$$

It is clear that a prescribed force leads to a Dirichlet-type boundary condition on the pressure. Similarly, a prescribed acceleration leads to a Neumann-type boundary condition for the pressure, which is just the product of Eq. (10) with the normal vector to the boundary. A combination of forces and motions therefore leads to a generalized Robin-type boundary condition.

Since Eq. (5) was formulated in the discrete setting, we will first discretize the fluid equations as well. Applying the symmetry-preserving finite-volume method (Verstappen and Veldman, 2003) to Eq. (10) with stress-free boundary conditions, we obtain the discrete momentum and continuity equations,

$$\frac{\Omega}{\Delta t}(\mathbf{u}_{i+1} - \mathbf{u}_i) + C(\mathbf{u}_i)\mathbf{u}_i + \frac{1}{\rho} G \mathbf{p}_{i+1} = \nu L \mathbf{u}_i + \Omega \mathbf{g}, \qquad G^T \mathbf{u}_{i+1} = 0. \tag{11}$$

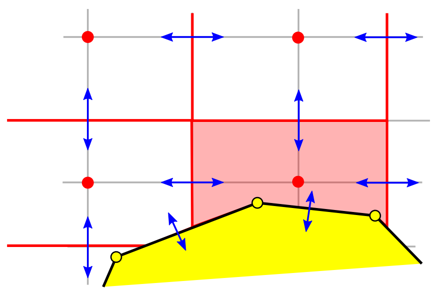

Fig. 3: Finite volume discretization on a staggered grid with cut cells. The solid is shown in yellow, pressures (and associated cells) in red, velocities in blue.

symbol	meaning	space
$\mathbf{p}$	pressure	$\mathbb{R}^k$
$\mathbf{u}$	velocity	$\mathbb{R}^{m+n}$
$\mathbf{g}$	gravity	$\mathbb{R}^{m+n}$
C	convection	$\mathbb{R}^{m+n} \to \mathbb{R}^{m+n}$
L	diffusion	$\mathbb{R}^{m+n} \to \mathbb{R}^{m+n}$
G	gradient	$\mathbb{R}^k \to \mathbb{R}^{m+n}$
G^T	divergence	$\mathbb{R}^{m+n} \to \mathbb{R}^k$
Ω	control volume	$\mathbb{R}^{m+n} \to \mathbb{R}^{m+n}$

Table 1: Discrete quantities

The pressure values are associated to cells, while the velocities are associated to cell faces, both internal and at the boundary, see Fig. 3. Considering a discrete domain with n boundary faces, m interior faces and k cells, the dimensions of the discrete unknowns and operators are defined in Table 1. We will focus on the n normal velocities that are defined at the fluid-solid boundary. In the original method (2), these velocities have to satisfy the motions of the solid. More precisely, at the boundary Γ the convective derivative of the fluid velocity follows the material acceleration of the solid

$$\frac{1}{\Delta t}\Gamma^T(\mathbf{u}_{i+1} - \mathbf{u}_i) = \ddot{\mathbf{x}}_i. \tag{12}$$

This constraint generates a force on the boundary, that is added as a Lagrange multiplier $\Gamma\mathbf{f}$ to the momentum equations. Since the velocity component normal to the boundary is considered, the forcing will also act normal to the boundary. Hence the fluid response $F(\ddot{\mathbf{x}})$ is the solution $\mathbf{f}_{i+1}$ of the following system

$$\begin{bmatrix} \frac{\Omega}{\Delta t} & G & \Gamma \\ G^T & 0 & 0 \\ \Gamma^T & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u}_{i+1} \\ \mathbf{p}_{i+1}/\rho \\ \mathbf{f}_{i+1}/\rho \end{bmatrix} = \begin{bmatrix} \frac{\Omega}{\Delta t} + \nu L - C & \Omega & 0 \\ 0 & 0 & 0 \\ \Gamma^T & 0 & \Delta t \end{bmatrix} \begin{bmatrix} \mathbf{u}_i \\ \mathbf{g} \\ \ddot{\mathbf{x}}_i \end{bmatrix}. \tag{13}$$

This system is usually solved by taking the Schur complement for the pressure, resulting in a discrete Poisson equation including the kinematic boundary condition Eq. (12). Instead, the proposed kinematic-dynamic boundary condition Eq. (4) requires only a small change to the fluid system. This change, indicated below in red, can be efficiently implemented as a modification of the discrete Poisson equation.

$$\begin{bmatrix} \frac{\Omega}{\Delta t} & G & \Gamma \\ G^T & 0 & 0 \\ \Gamma^T & 0 & -\rho\Delta t D \end{bmatrix} \begin{bmatrix} \mathbf{u}_{i+1} \\ \mathbf{p}_{i+1}/\rho \\ \mathbf{f}_{i+1}/\rho \end{bmatrix} = \begin{bmatrix} \frac{\Omega}{\Delta t} + \nu L - C & \Omega & 0 \\ 0 & 0 & 0 \\ \Gamma^T & 0 & \Delta t \end{bmatrix} \begin{bmatrix} \mathbf{u}_i \\ \mathbf{g} \\ \ddot{\mathbf{x}}_i \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ \Delta t D \mathbf{f}_i \end{bmatrix}. \tag{14}$$

$$\frac{1}{\Delta t}\Gamma^T(\mathbf{u}_{i+1} - \mathbf{u}_i) = \ddot{\mathbf{x}}_i + D(\mathbf{f}_{i+1} - \mathbf{f}_i). \tag{15}$$

When the stability assumptions of section 4 are satisfied, this approach may be used as a weak coupling.

Results

The original (2) and modified (5) iterations have been applied to a simple two-dimensional flow around a cylinder. The cylinder is attached to a spring system and is allowed to interact freely with the flow. The added mass of a circular cylinder in an incompressible flow is known to be equal to the displaced mass of fluid. By varying the mass of the cylinder, a range of added mass ratios is obtained between 0.1 and 50. The approximate dynamics D are based on the exact mass of the cylinder.

To obtain a fair comparison of the performance of both methods, the total computational cost per timestep should be compared. However, the original and modified methods solve a different set of equations within each fluid-solid iteration, hence the amount of work per iteration is not equal. Therefore we report the total number of matrix-vector products summed over all evaluations of the fluid problem within a timestep. This accurately represents the relative performance of both methods. Since the bulk of the work is commonly in the matrix-vector products, it is also a measure of their absolute performance. The modified method performs superior to the original method for all added mass ratios even below one. When the approximate dynamics are based on the exact mass (without stiffness and damping), the performance becomes independent of the added mass ratio.

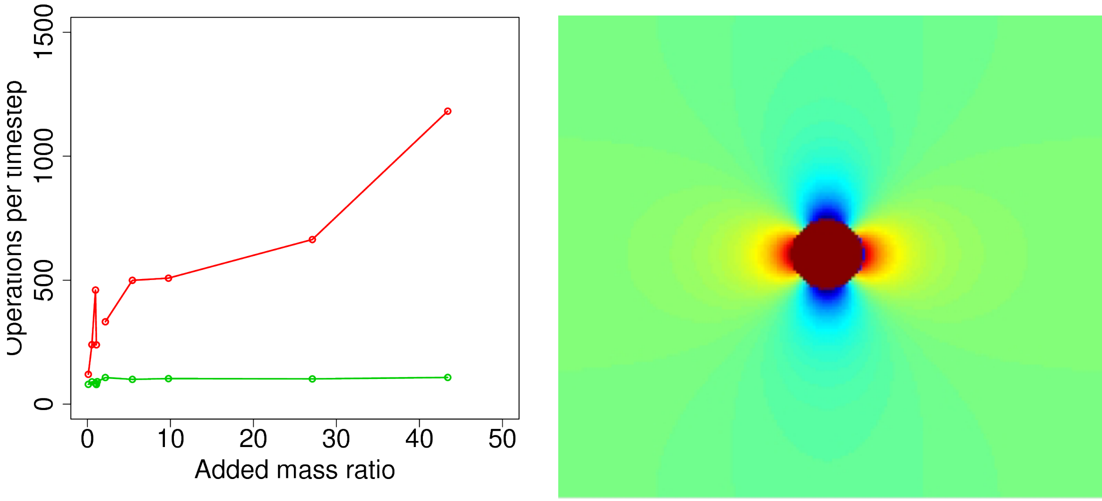

Fig. 4: Total number of matrix-vector products per timestep against added mass ratio of a submerged cylinder. Original method shown in red, the current modified method shown in green.

Recommendations

In a partitioned fluid-solid coupling with an incompressible fluid, the ratio of added mass to structural mass turn out to have an important effect on the coupling efficiency. Especially for applications involving either floating structures, objects crossing the free surface or fluids in confined geometries, large added mass ratios are observed.

In these situations, it is advisable to modify the coupling iteration to allow the fluid to respond better to the solid motions. A relatively simple change in the fluid boundary condition, only requiring knowledge of the mass of the structure can provide a large performance increase.

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