# ON CONSERVATIVE SMOOTHERS FOR TURBULENT CONVECTION: AN ALTERNATIVE SIMULATION SHORTCUT

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### ABSTRACT

Most turbulent flows can not be computed directly from the Navier-Stokes equations, because they possess far too many scales of motion. The computationally almost numberless small scales result from the nonlinear convective term which allows for the transfer of energy from scales as large as the flow domain to the smallest scales that can survive viscous dissipation. In the quest for a simulation shortcut, we propose to smooth the convective term in such a way that the symmetries that yield the invariance of the energy, enstrophy (in 2D) and helicity are preserved. This requirement yields a class of conservative smoothers. The numerical algorithm used to solve the governing equations also preserves the symmetries and is therefore well-suited to test the proposed simulation shortcut. The simulation shortcut is successfully tested for turbulent channel flow ($\mathrm{Re}_\tau = 180$).

### NAVIER-STOKES EQUATIONS AND TURBULENCE

The Navier-Stokes equations provide an appropriate model for turbulent flow. In the absence of compressibility ($\nabla \cdot u = 0$), the equations are

$$\partial_t u + \mathcal{C}(u,u) + \mathcal{D}(u) + \nabla p = 0 \qquad (1)$$

where the nonlinear, convective term is given by

$$\mathcal{C}(u,v) = (u \cdot \nabla)v$$

and the dissipative term reads $\mathcal{D}(u) = -\Delta u/\mathrm{Re}$; the parameter Re is the Reynolds number. At the present levels of computing power, attempts at simulating turbulence directly from (1) are limited to 'a milli-second over a postage stamp' (Spalart, 2000), because the solution of Eq. (1) possesses far too many scales of motion. The computationally almost numberless small scales result from the nonlinear, convective term which allows for the transfer of energy from scales as large as the flow domain to the smallest scales that can survive viscous dissipation. As the full energy cascade can not be computed, a dynamically less complex mathematical formulation is sought.

In the quest for such a formulation, the Navier-Stokes equations may be filtered spatially like in large eddy simulation (LES). In LES, the nonlinear, filtered terms involving small-scale motions are to be modelled to arrive at the following equation governing the large eddies:

$$\partial_t \overline{u}_\epsilon + \mathcal{C}(\overline{u}_\epsilon, \overline{u}_\epsilon) + \mathcal{D}(\overline{u}_\epsilon) + \nabla \overline{p}_\epsilon = \mathrm{model}(\overline{u}_\epsilon) \qquad (2)$$

Here, the filtered velocity is denoted by $\overline{u}$ and the variable name is changed from $u$ to $u_\epsilon$ to stress that the solution of (2) differs from that of (1). The closure model in the right-hand side of (2) approximates the commutator of $\mathcal{C}$ and the filter

$$\mathrm{model}(\overline{u}_\epsilon) \approx \mathcal{C}(\overline{u}_\epsilon, \overline{u}_\epsilon) - \overline{\mathcal{C}(u_\epsilon, u_\epsilon)} \qquad (3)$$

An appropriate closure model is hard to accomplish for a number of reasons. Carati et al. (1999) have shown that for all symmetric filters (that are $C^\infty$ in wave space) the series expansion of the commutator (in powers of the filter length) starts with a term that is known under various names, among others nonlinear model, gradient model and tensor-diffusivity model. This generic, leading-order term turns out to give rise to severe instabilities (Vreman, 1995). In practice, closure models are often based on phenomenological arguments that can not be derived formally from the Navier-Stokes equations. Sagaut (2001), e.g., gives an overview of closure models; for a recent review on mathematical issues related to the theory of LES, see Guermond et al. (2004).

Throughout this paper, we consider smooth approximations (regularizations) of the convective term in (1):

$$\partial_t u_\epsilon + \widetilde{\mathcal{C}}(u_\epsilon, u_\epsilon) + \mathcal{D}(u_\epsilon) + \nabla p_\epsilon = 0 \qquad (4)$$

This approach falls in with the concept of LES if the approximate, convective term is taken such that

$$\overline{\widetilde{\mathcal{C}}(u_\epsilon, u_\epsilon)} = \mathcal{C}(\overline{u}_\epsilon, \overline{u}_\epsilon) - \mathrm{model}(\overline{u}_\epsilon) \qquad (5)$$

Indeed under this condition, Eq. (4) is equivalent to (2): we can filter (4) first and thereafter compare the filtered version of (4) term-by-term with (2) to identify the closure model. Notice that (5) is satisfied for any invertible filter, e.g. holds for a box, Gaussian or Helmholtz filter, etc. Additionally, it may be observed that the combination (3)+(5) gives

$$\overline{\widetilde{\mathcal{C}}(u_\epsilon, u_\epsilon)} \approx \overline{\mathcal{C}(u_\epsilon, u_\epsilon)} \qquad (6)$$

which states that at least the low modes of $\mathcal{C}$ are to be approximated consistently.

The idea is to smooth the convective term directly to set bounds to the creation of smaller and smaller scales of motion and thus to confine the cascade of energy. To that end, we need not consider an equation for the filtered velocity. It suffices that the low modes of the solution $u_\epsilon$ of Eq. (4) approximate the corresponding low modes of the solution $u$ of the Navier-Stokes equations (1), whereas the high modes of $u_\epsilon$ vanish much faster than those of $u$. In that case, Eq. (4) provides a basis for a simulation shortcut.

The first outstanding approach in this direction goes back to Leray (1934), who took $\widetilde{\mathcal{C}}(u,u) = \mathcal{C}(\bar{u}, u)$ and proved that a moderate filtering of the convective velocity is sufficient to regularize a turbulent flow. Cheskidov et al. (2005) have analyzed Leray's approximation for a Helmholtz filter. They show that the complexity of the 3D Leray model lies between that of the 2D and 3D Navier-Stokes equations. The Navier-Stokes-alpha-model forms another example of regularization modeling (Holm et al., 1998 and Foias et al., 2001). In this model, the convective term becomes $\widetilde{\mathcal{C}}_r(u,u) = \mathcal{C}_r(u,\bar{u})$, where $\mathcal{C}_r$ denotes the convective operator in rotational form: $\mathcal{C}_r(u,v) = (\nabla \times u) \times v$.

The regularization method basically alters the nonlinearity to control the convective energetic exchanges. In doing so, one can preserve certain fundamental properties of (the convective operator in) the Navier-Stokes equations exactly, e.g., symmetries, conservation properties, transformation properties, Kelvin's circulation theorem, Bernouilli's theorem, Karman-Howarth theorem, etc. Of course, any of these properties holds approximately, where the error depends upon the considered approximation of $\mathcal{C}$.

In this paper, we propose to approximate $\mathcal{C}$ in such a manner that the symmetry properties that yield the invariance of the energy, the enstrophy (in 2D) and helicity (in 3D) are preserved. The underlying idea is to restrain the convective production of smaller and smaller scales of motion by means of vortex stretching, while ensuring that the solution does not blow up (in the energy-norm; in 2D also: enstrophy-norm). We anticipate that the unconditional stability enhances the accuracy at coarse resolutions. Here, it may be stressed the unconditional stability of $\widetilde{\mathcal{C}}$ allows for simulations at arbitrary coarse grids, provided the numerical approximation of $\widetilde{\mathcal{C}}$ preserves the symmetry too (Verstappen and Veldman, 2003).

## SYMMETRY AND CONSERVATION PROPERTIES

To sketch the symmetry and conservation properties of turbulent convection, we introduce the concepts of energy, enstrophy and helicity. In terms of the usual scalar product $(u,v) = \int_\Omega u \cdot v dx$, the energy of a fluid with velocity $u$ and occupying a region $\Omega$ is given by $|u|^2 = (u,u)$. The enstrophy is defined as $|\nabla \times u|^2$ and the helicity is given by $(\nabla \times u, u)$. In the computations that come, we restrict ourselves to solenoidal fields and ignore all contributions resulting from boundary conditions, i.e., we consider either the whole space with velocities vanishing sufficiently rapidly at infinity, or a finite domain with periodic conditions (or in general, with boundary conditions sufficient for the conservation properties).

The evolution of the energy follows from differentiating $(u,u)$ with respect to time and rewriting $\partial_t u$ with the help of (1). In this way, we get a convective contribution given by the trilinear form $(\mathcal{C}(u,u),u)$. Since this form is skew-symmetry with respect to the last two arguments, that is

$$(\mathcal{C}(u,v),w) = -(v,\mathcal{C}(u,w)) \tag{7}$$

(see e.g. Temam, 1995), we have $(\mathcal{C}(u,v),v) = 0$ for any pair $u, v$, which implies that the convective contribution $(\mathcal{C}(u,u),u)$ cancels from the energy equation; hence the energy is conserved in the absence of viscous dissipation ($\mathcal{D} = 0$).

The evolution of the enstrophy can be obtained by taking the inner product of the Navier-Stokes equations with the vector field $-\Delta u$. The resulting convective contribution vanishes in two spatial dimensions (Temam, 1995):

$$(\mathcal{C}(u,u),\Delta u) = 0 \tag{8}$$

Actually, an even stronger form of enstrophy invariance holds (Vukadinovic, 2004):

$$(\mathcal{C}(u,v),\Delta v) = (u,\mathcal{C}(\Delta v,v)). \tag{9}$$

Recalling that according to Eq. (7) $(u,\mathcal{C}(\cdot,u)) = 0$, we see that Eq. (9) with $u = v$ implies (8).

The evolution of the helicity follows from the inner product of Eq. (1) with the vorticity $\omega = \nabla \times u$ and the inner product of the vorticity equation

$$\partial_t \omega + \mathcal{C}(u,\omega) + \mathcal{D}(\omega) = \mathcal{C}(\omega,u), \tag{10}$$

with the velocity $u$. Taking these inner products yields the convective contribution $(\mathcal{C}(u,u),\omega)+(\mathcal{C}(u,\omega),u)-(\mathcal{C}(\omega,u),u)$, which vanishes as an immediate consequence of the skew-symmetry (7). Therefore, the helicity is conserved in the absence of viscous dissipation.

## A CLASS OF CONSERVATIVE SMOOTHERS

Approximations of particular interest are the ones that preserve the energy, the enstrophy (in 2D) and the helicity in the absence of viscous dissipation, among others because they are intrinsically stable (in the energy-norm; in 2D also: enstrophy-norm). The Leray model conserves the energy, but not the enstrophy or helicity, whereas the Navier-Stokes-alpha model conserves the enstrophy and helicity, yet not the energy.

In the previous section, we saw that the conservation of the energy, enstrophy and helicity is intimately tied up with the symmetry properties of the convective operator $\mathcal{C}$. Therefore, we aim to approximate $\mathcal{C}$ in such manner that the underlying symmetries (given by Eq. (7) and Eq. (9)) are preserved. This criterion yields the following class of approximations

$$\partial_t u_\epsilon + \mathcal{C}_n(u_\epsilon,u_\epsilon) + \mathcal{D}(u_\epsilon) + \nabla p_\epsilon = 0 \tag{11}$$

($n = 2,4,6$) in which the convective term is smoothened according to

$$\mathcal{C}_2(u,v) = \overline{\mathcal{C}(\overline{u},\overline{v})} \tag{12}$$

$$\mathcal{C}_4(u,v) = \mathcal{C}(\overline{u},\overline{v}) + \overline{\mathcal{C}(\overline{u},v')} + \overline{\mathcal{C}(u',\overline{v})} \tag{13}$$

$$\mathcal{C}_6(u,v) = \mathcal{C}(\overline{u},\overline{v}) + \mathcal{C}(\overline{u},v') + \mathcal{C}(u',\overline{v}) + \overline{\mathcal{C}(u',v')} \tag{14}$$

Here a bar denotes a filtered quantity and a prime indicates the residual.

The three approximations $\mathcal{C}_n(u,u)$ are consistent with $\mathcal{C}(u,u)$, where the error is of the order of $\epsilon^n$ with $n = 2,4,6$ for symmetric filters with a filter length $\epsilon$. Both the Leray model and the alpha model are second-order accurate in terms of $\epsilon$.

The approximations (12)-(14) are constructed in such a way that the fundamental properties (7) and (9) are preserved. That is, it can be shown that $\mathcal{C}_n$ inherits the skew-symmetry of $\mathcal{C}$: for any filter satisfying $(\overline{u},v) = (u,\overline{v})$,

$$(\mathcal{C}_n(u,v),w) = -(v,\mathcal{C}_n(u,w)) \tag{15}$$

where $n = 2,4,6$, and in 2D (provided the filter commutes with differentiation):

$$(\mathcal{C}_n(u,v),\Delta v) = (u,\mathcal{C}_n(\Delta v,v)) \tag{16}$$

Consequently, the evolution of the energy $\frac{1}{2}|u_\epsilon^2|$ of any solution $u_\epsilon$ of (11)-(14) is given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}|u_\epsilon|^2 = -\frac{1}{\mathrm{Re}}|\nabla u_\epsilon|^2 = -\frac{1}{\mathrm{Re}}|\nabla \times u_\epsilon|^2 \tag{17}$$

where the second equality can be established by means of integration by parts (using the divergence-free condition and ignoring boundary terms). Hence, the enstrophy determines the rate of dissipation of energy.

The enstrophy equation maintains its well-known structure:

$$\frac{\mathrm{d}}{\mathrm{d}t}\tfrac{1}{2}|\nabla\times u_\epsilon|^2 = -\tfrac{1}{\mathrm{Re}}|\Delta u_\epsilon|^2 - (\mathcal{C}_n(u_\epsilon,u_\epsilon),\Delta u_\epsilon) \quad (18)$$

The enstrophy of an inviscid flow governed by (11)-(14) is conserved in two spatial dimensions because (15)-(16) result into $(\mathcal{C}_n(u_\epsilon,u_\epsilon),\Delta u_\epsilon)=0$. In 3D, the notorious high fluctuations of the trilinear form are damped, by replacing $(\mathcal{C}(u,u),\Delta u)$ by $(\mathcal{C}_n(u_\epsilon,u_\epsilon),\Delta u_\epsilon)$, in the hope that this smoothing prevents the vorticity from bursting to very small scales. The rigorous mathematical analysis of the regularity of the solution does not fall within the scope of the present paper. Perhaps, the analysis may be performed by means of the mathematical techniques that have been applied to prove the regularity of Leray and Navier-Stokes-alpha solutions (see Cheskidov et al. (2005) and Foias et al. (2001), respectively).

By taking the curl of (11), we obtain the evolution of the vorticity $\omega_\epsilon = \nabla\times u_\epsilon$:

$$\partial_t\omega_\epsilon + \mathcal{C}_n(u_\epsilon,\omega_\epsilon) + \mathcal{D}(\omega_\epsilon) = \mathcal{C}_n(\omega_\epsilon,u_\epsilon) \quad (19)$$

where $n=2,4,6$ (provided the filter commutes with differentiation). This evolution equation resembles the vorticity equation (10): the only difference is that $\mathcal{C}$ is replaced by the approximation $\mathcal{C}_n$. Now, as $\mathcal{C}$ and $\mathcal{C}_n$ possess the same symmetry (meaning that both $(\mathcal{C}(u,v),w)$ and $(\mathcal{C}_n(u,v),w)$ are skew-symmetric with respect to their last two arguments), an argument similar to the one used to establish the conservation of helicity in the previous section, tells us that the helicity is conserved if the approximation $\mathcal{C}_n$ is applied.

## VORTEX STRETCHING MECHANISM

If it happens that the source term $\mathcal{C}_n(\omega_\epsilon,u_\epsilon)$ in Eq. (19) is so strong that the dissipative term $\mathcal{D}(\omega_\epsilon)$ can not prevent the intensification of vorticity, smaller and smaller vortical structures may be produced locally. The Navier-Stokes equations give the source term

$$\mathcal{C}(\omega,u) = S\omega = \overline{S}\overline{\omega} + \overline{S}\omega' + S'\overline{\omega} + S'\omega' \quad (20)$$

where $S=\frac{1}{2}(\nabla u + \nabla u^{\mathrm{T}})$ is the deformation tensor. The trace of this symmetric tensor is zero. Hence, $S$ has at least one non-negative eigenvalue. If $\omega$ is aligned with an eigenvector associated with a positive eigenvalue, then the source term $\mathcal{C}(\omega,u)$ in Eq. (10) is positive, which may lead to an increase of the vorticity magnitude. As the angular momentum is conserved (in the absence of viscous dissipation) an increase of the vorticity magnitude implies that fluid elements are stretched along the direction of the eigenvector associated with the positive eigenvalue. This phenomenon, called vortex stretching, implies a transfer of energy from large scales of motion to smaller ones, i.e., drives the energy cascade. The conservative approximations (12)-(14) change this transfer of energy in three spatial dimensions (in 2D the vortex stretching term $\mathcal{C}_n(\omega_\epsilon,u_\epsilon)$ is identically zero, because the vorticity has a nonzero component only in the direction where the velocity is zero). The vortex stretching term $\mathcal{C}_n(\omega_\epsilon,u_\epsilon)$ in Eq. (19) reads

$$\mathcal{C}_2(\omega,u) = \overline{\overline{S}\overline{\omega}} \quad (21)$$

$$\mathcal{C}_4(\omega,u) = \overline{S}\overline{\omega} + \overline{\overline{S}\omega'} + \overline{S'\overline{\omega}} \quad (22)$$

$$\mathcal{C}_6(\omega,u) = \overline{S}\overline{\omega} + \overline{S}\omega' + S'\overline{\omega} + \overline{S'\omega'} \quad (23)$$

Qualitatively, vortex stretching leads to the production of smaller and smaller scales; hence to a continuous, local increase of both $S'$ and $\omega'$. Consequently, at the positions where vortex stretching occurs, the terms with $S'$ and $\omega'$ will eventually amount considerably to the right-hand side of (20). Since these terms are diminished in (21)-(23), the conservative smoothing of the convective term counteracts the production of smaller and smaller scales by means of vortex stretching and may eventually stop the continuation of the vortex stretching process.

## TRIADIC INTERACTIONS

To study the interscale interactions in more detail, we continue in the spectral space, where we will restrict ourselves to the approximation $\mathcal{C}_4$; a similar analysis can be performed for $\mathcal{C}_2$ or $\mathcal{C}_6$. The spectral representation of the convective term in the Navier-Stokes equations is given by $\mathcal{C}_k(\hat{u},\hat{u})$ with

$$\mathcal{C}_k(\hat{u},\hat{v}) = \mathrm{i}\Pi(k)\sum_{p+q=k}\hat{u}_p q\hat{v}_q \quad (24)$$

where $\Pi(k)=I-kk^{\mathrm{T}}/|k|^2$ denotes the projector onto divergence free velocity fields in the spectral space. Taking the Fourier transform of (11)+(13), we obtain the evolution of each Fourier-mode $\hat{u}_k(t)$ of $u_\epsilon$ for the approximation $\mathcal{C}_4$:

$$\left(\frac{\mathrm{d}}{\mathrm{d}t} + \frac{|k|^2}{\mathrm{Re}}\right)\hat{u}_k + \mathcal{C}_{4,k}(\hat{u},\hat{u}) = 0 \quad (25)$$

where $\mathcal{C}_{4,k}(\hat{u},\hat{v})$ equals

$$\mathcal{C}_k(\hat{G}\hat{u},\hat{G}\hat{v}) + \hat{G}_k\mathcal{C}_k(\hat{G}\hat{u},(I-\hat{G})\hat{v}) + \hat{G}_k\mathcal{C}_k((I-\hat{G})\hat{u},\hat{G}\hat{v})$$

and $\hat{G}$ denotes the Fourier transform of the symmetric kernel of our generic convolution filter. The Navier-Stokes dynamics is obtained if $\hat{G}=I$.

The mode $\hat{u}_k(t)$ interacts only with those modes whose wavevectors $p$ and $q$ form a trangle with the vector $k$. Modes for which $\epsilon|k|<1$ are largely unaffected by the filter. Indeed, a generic, symmetric convolution filter satisfies

$$\hat{G}_k = 1-\epsilon^2|k|^2/24 + \mathrm{h.o.t.}$$

(see for instance Caratti et al., 1999). Consequently, for any triangle $k=p+q$ satisfying $\epsilon|k|<1$, $\epsilon|p|<1$ and $\epsilon|q|<1$, the leading-order terms of $\mathcal{C}_{4,k}(\hat{u},\hat{v})$ become

$$\mathrm{i}\Pi(k)\sum_{p+q=k}\hat{u}_p q\hat{v}_q\left(1-\frac{\epsilon^4}{24^2}(|k|^2|p|^2+|k|^2|q|^2+|p|^2|q|^2)\right)$$

In conclusion, the local interactions between large scales of motion (meaning that $\epsilon|k|<1$ and $|k|\sim|p|\sim|q|$) approximate the Navier-Stokes dynamics up to fourth order. In other words, the triadic interactions between large scales are only slightly altered by the approximation $\mathcal{C}_4$, which means that $\mathcal{C}_4$ satisfies the consistency condition given by Eq. (6).

In order to investigate interactions involving longer wavevectors (smaller scales of motion), the filter need be specified further. Since a Helmholtz filter enables a plain analysis of the interactions, we consider

$$\hat{G}_k = \frac{1}{1+\alpha^2|k|^2}$$

where $\alpha^2 = \epsilon^2/24$. For this filter, the approximation $\mathcal{C}_{4,k}$ becomes

$$\mathrm{i}\Pi(k) \sum_{p+q=k} \hat{u}_p q \hat{v}_q \frac{1+\alpha^2(|k|^2+|p|^2+|q|^2)}{(1+\alpha^2|k|^2)(1+\alpha^2|p|^2)(1+\alpha^2|q|^2)}$$

By comparing this expression with Eq. (24), we see that all contributions to the sum are reduced by the application of the Helmholtz filter. The amount by which the interactions are lessened depends on the length of the legs of the triangle $k = p+q$. The reduction is the largest for triangles with three long legs, i.e. $\alpha|k| > 1$, $\alpha|p| > 1$ and $\alpha|q| > 1$. For those triangles, the long-leg limit of the reduction factor is given by

$$\frac{1}{\alpha^4|k|^2|p|^2} + \frac{1}{\alpha^4|k|^2|q|^2} + \frac{1}{\alpha^4|p|^2|q|^2}$$

The nonlocal interactions between a large scale characterized by $\alpha|k| < 1$ and two small scales satisfying $\alpha|p| > 1$ and $\alpha|q| > 1$ are (for $|k| \ll |p|, |q|$) about

$$\frac{1}{\alpha^2|p|^2} + \frac{1}{\alpha^2|q|^2}$$

times less when the approximation $\mathcal{C}_4$ is used instead of $\mathcal{C}$. In general, we see that with a Helmholtz filter the approximation $\mathcal{C}_4$ (strongly) attenuates all interactions for which at least two legs of the triangle $k = p+q$ are (much) longer than $1/\alpha$, whereas all possible triadic interactions for which at least two legs are (much) shorter than $1/\alpha$ are reduced to a small degree. Notice that in the latter case the longest leg is always shorter than $2/\alpha$. Thus, we may conclude that the approximation $\mathcal{C}_4$ confines the dynamics for the greatest part to scales whose wavevector-length is smaller than $2/\alpha$. In this way, the resolution requirements resulting from the convective nonlinearity are reduced and may become within the reach of present day simulation techniques.

## BERNOULLI'S THEOREM

It may be observed that in (1) no distinction is made between the convective velocity and the actual fluid velocity: the nonlinearity may also be represented by $\mathcal{N}(u) = \mathcal{C}(u,u)$. The same holds for the conservative models which smooth $\mathcal{N}(u)$ to $\mathcal{N}_n(u) = \mathcal{C}_n(u,u)$. Consequently, a number of transport properties of convection can be translated straightforwardly to (11)-(14). As an example, we note that Bernoulli's theorem changes from its well-known form to: in a stationary ($\partial_t u = 0$), inviscid ($\mathcal{D} = 0$) flow governed by (11)+(13) the quantity

$$\tfrac{1}{2}\overline{|u_\epsilon|^2} + \tfrac{1}{2}\left(|\overline{u}_\epsilon|^2\right)' - \tfrac{1}{2}\overline{|u'_\epsilon|^2} + p_\epsilon$$

is constant along any curve parameterized by $r(t)$ where the tangential vector satisfies

$$\dot{r} \perp \overline{(\nabla \times u_\epsilon) \times u_\epsilon} + ((\nabla \times \overline{u}_\epsilon) \times \overline{u}_\epsilon)' - \overline{(\nabla \times u'_\epsilon) \times u'_\epsilon}$$

The proof uses the same lines of reasoning as the proof of Bernoulli's theorem by Chorin and Marsden (1979) and is therefore omitted here. Similar results can be proven for $n = 2$ and $n = 6$. Hence, (11)-(14) are not only conservative, but also allow for the definition of smooth streamlines along which a smooth form of $\frac{1}{2}|u|^2 + p$ is transported.

## NUMERICAL SIMULATION METHOD

The smooth approximations $\mathcal{C}_n$ given by Eqs. (12)-(14) are constructed such that fundamental properties (7) and (9) are preserved. Of coarse, the same should hold for the numerical approximations that are used to discretize $\mathcal{C}_n$. Therefore, we have developed spatial discretizations of $\mathcal{C}_n$ which preserve the properties (15) and (16). In short (for a detailed explanation, see Verstappen and Veldman, 2003), the temporal evolution of the spatially discrete velocity vector $u_h$ is governed by the following fourth-order, finite-volume discretization of Eq. (11)

$$\Omega \frac{\mathrm{d}u_h}{\mathrm{d}t} + C_n(u_h)\,u_h + Du_h - M^{\mathrm{T}} p_h = 0 \qquad (26)$$

where the incompressibility constraint reads $Mu_h = 0$. The diffusive matrix $D$ is symmetric and positive-definite; the diagonal matrix $\Omega$ represents the sizes of the control volumes. The convective matrix $C_n(u_h)$ is skew-symmetric,

$$C_n(u_h) + C_n^{\mathrm{T}}(u_h) = 0 \qquad (27)$$

and is constructed such that enstrophy-invariance (8) holds discretely (in 2D). Eq. (27) forms the discrete analogon of Eq. (15). In discrete form, the skew-symmetry reads: for all discrete velocity-vectors $u_h$, $v_h$ and $w_h$, we have

$$C_n(u_h)\,v_h \cdot w_h = v_h \cdot C_n^{\mathrm{T}}(u_h)\,w_h \overset{(27)}{=} -v_h \cdot C_n(u_h)\,w_h$$

The evolution of the discrete energy $||u_h||^2 = u_h \cdot \Omega u_h$ of any solution $u_h$ of (26) is governed by

$$\frac{\mathrm{d}}{\mathrm{d}t}||u_h||^2 = -u_h \cdot (D + D^{\mathrm{T}})u_h \leq 0 \qquad (28)$$

where the convective contribution cancels because of (27); compare the discretization of the energy equation given by (28) with continuous expression (17). Note that the rate of change of energy is neither influenced by pressure differences since $Mu_h = 0$. So, in conclusion, the discrete energy does not increase; hence the spatial discretization is stable on any grid. Therefore, the choice of the grid may be based on the required accuracy solely and the main question becomes: how accurate is the model given by (26).

## FIRST RESULTS FOR TURBULENT CHANNEL FLOW

As a first step in application of the conservative smoothers, the approximation $\mathcal{C}_4$ is tested for a turbulent channel flow by means of a comparison with the direct numerical simulations performed by Kim et al. (1987). Based on the channel half-width and the bulk velocity the Reynolds number is 2800. This flow forms a prototype for near-wall turbulence. We consider two, coarse, computational grids consisting of 16×16×8 and 32×32×16 grid points, respectively. More details about the numerics (grid-stretching, time-stepping, etc.) can be found in Verstappen and Veldman (2003).

The filtering is based upon the Helmholtz operator, where the boundary conditions that supplement the Navier-Stokes equations are applied to the filter too. Since the computation of the numerical solution of the Helmholtz equation for $\overline{u}$ is

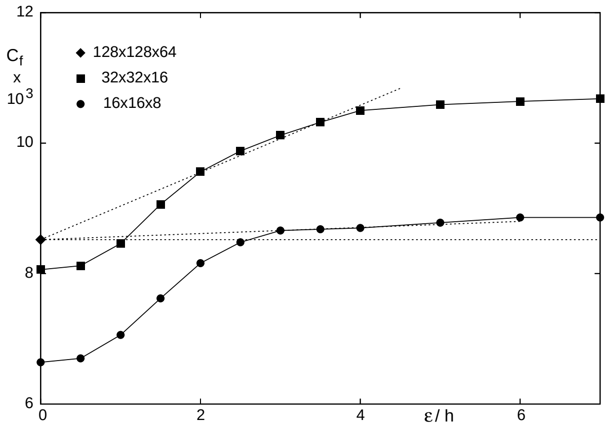


Figure 1: The skin friction coefficient as function of $\epsilon/h$. The results are computed with the help of the approximation $\mathcal{C}_4$. The symbols represent the data at three different grids.

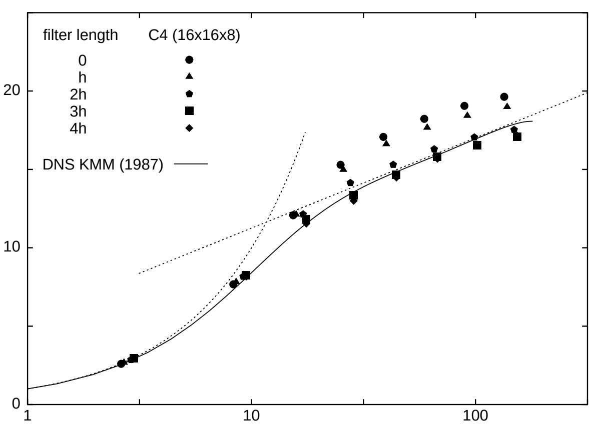


Figure 2: The mean velocity (in wall coordinates) as obtained from the 16×16×8 simulations. The filter length $\epsilon$ varies from zero to four times the grid width $h$.

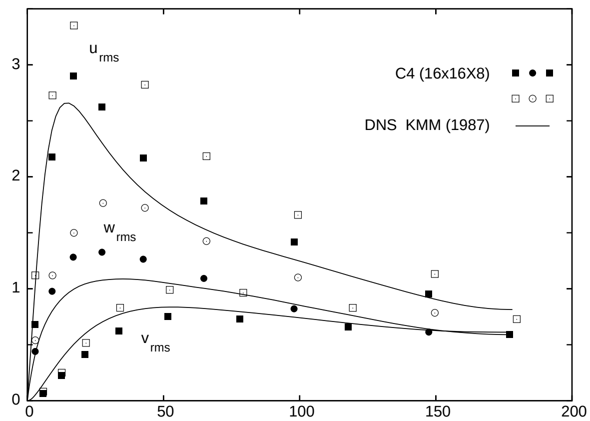


Figure 3: The root-mean-square velocity fluctuations ($\mathcal{C}_4$; 16×16×8 grid points). The open boxes and circles correspond with $\epsilon = 2.5h$; the filled boxes and circles represent data that is extrapolated to $\epsilon = 0$.

rather expensive, we note that the filter serves as a smoothing operation only. Therefore, we do not fully solve the Helmholtz equation for $\overline{u}$, but choose to perform just one Jacobi iteration with $\overline{u} = u$ as initial guess. Here, the filter length $\epsilon$ can take any value, i.e., need not be restricted to integer multiples of the grid width.

Figure 1 shows the skin friction coefficient as function of the ratio of the filter length $\epsilon$ to the grid width $h$ (for the approximation $\mathcal{C}_4$). The reference value $C_f = 8.52\times10^{-3}$ is computed from a simulation with 128×128×64 grid points and $\epsilon = 0$ (that is, without any models). This value is in good agreement with Dean's correlation of $C_f = 0.073\ Re_m^{-0.25} = 8.44\times10^{-3}$ (Dean, 1976). We see that the 32×32×16 calculations predict the reference value of the skin friction well if the filter length $\epsilon$ is taken (approximately) equal to the grid width $h$. For the 16×16×8 grid, the optimal ratio of $\epsilon$ to $h$ is about 2.5. Additionally, it may be observed that the reference value of the skin friction can be estimated accurately by extrapolating from greater filter-lengths. In case of the 32×32×16 grid, a linear extrapolation from values within the range $2 < \epsilon/h < 4$ provides the reference value of $C_f$ at $\epsilon = 0$; for the 16×16×8 grid the estimated value at $\epsilon = 0$ is to be based upon $3 < \epsilon/h < 5$. This suggests that we may possibly undo the smoothing with a (properly chosen) linear extrapolation. Here, we have to extrapolate the simulation results to $\epsilon = 0$ (no model), because the simulations are underresolved if the grid is coarse and the filter has little length. Therefore, we should not consider the 16×16×8 and 32×32×16 results for small $\epsilon$.

The least to be expected from a simulation shortcut is a good prediction of the mean flow. As can be seen in Fig. 2, the approximation $\mathcal{C}_4$ satisfies that minimal requirement already at the very coarse 16×16×8 grid, provided the filter length $\epsilon$ is taken in the range ($2h$, $4h$). Yet, the turbulence intensities do not agree so well with the reference data if only 2048 gridpoints are used (Fig. 3). Here, it may be noted that the root-mean-square velocity fluctuations are the least worse (in comparison to the DNS of Kim et al. (1987)) if the results are extrapolated linearly to $\epsilon = 0$. Again, this observation indicates that a linear extrapolation from sufficiently large $\epsilon$ to $\epsilon = 0$ may possibly undo the effect of the smoothing.

Overall good agreement between the $\mathcal{C}_4$-calculation at the 32×32×16 grid and the DNS is observed for both the first- and second-order statistics, see Fig. 4 and Fig. 5.

Heuristic arguments as well as computational results (Fig. 6-7) show that the energy spectrum of the solution of (11)+(13) follows the DNS for large scales of motion, whereas a much steeper (numerically speaking: more gentle) power law is found for small scales. The smoothing of $\mathcal{C}$ improves the agreement with the DNS for the low modes, whereas the high modes vanish faster, which is precisely what a simulation shortcut is ought to do.

The first results shown here illustrate the potential of conservative smoothing as a new simulation shortcut for turbulent channel flow. Yet, given the inherent difficulty of turbulence modeling, more thorough investigations and comparisons need be carried out to clarify the pros and cons of conservative smoothing.

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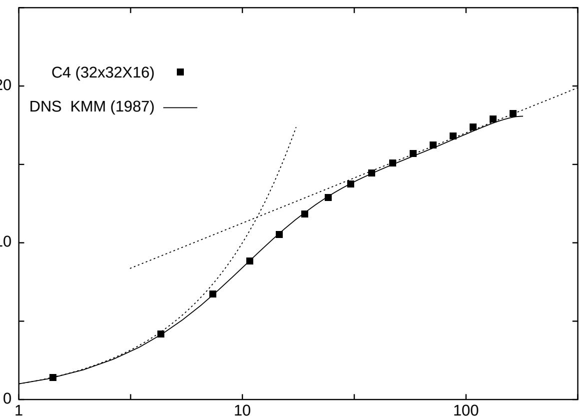


Figure 4: The mean velocity (in wall coordinates) of the $32{\times}32{\times}16$ simulations for $\epsilon = h$.

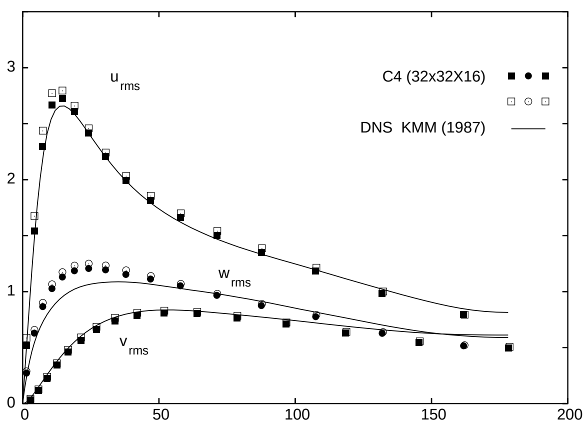


Figure 5: Rms velocity fluctuations; $32{\times}32{\times}16$ grid points. The open boxes and circles correspond with $\epsilon = h$; the filled boxes and circles represent data that is extrapolated to $\epsilon = 0$.

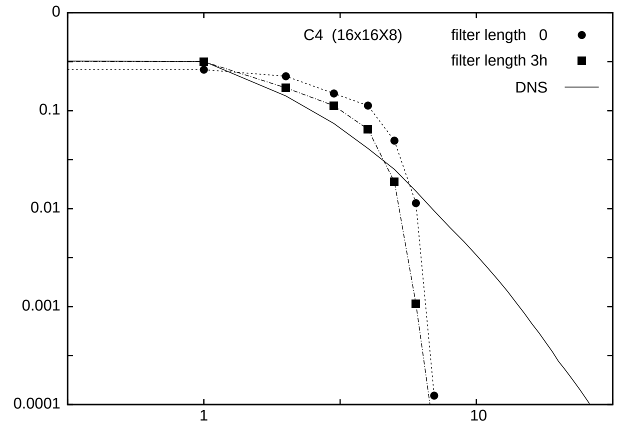


Figure 6: One-dimensional (streamwise) energy spectrum at $y^+ \approx 5$, i.e. at the first grid point of the $16{\times}16{\times}8$ grid.

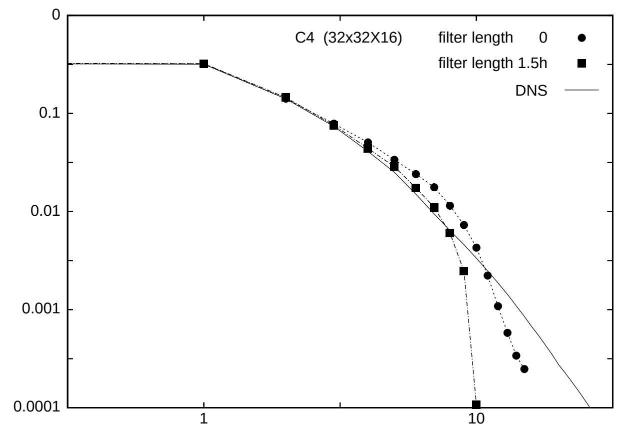


Figure 7: One-dimensional (streamwise) energy spectrum at $y^+ \approx 5$; $32{\times}32{\times}16$ grid points. The smoothing improves the low modes, whereas the high modes vanish faster.

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